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Superembedding methods for 4d \mathcal{N} -extended SCFTs

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Abstract

We consider the embedding method of the superconformal group in four dimensions in the case of extended supersymmetry, hence generalizing the recent work of Goldberger, Skiba and Son which was restricted at $\mathcal{N} = 1$. Moreover, we work out explicitly the case of $\mathcal{N} = 2$ chiral superfields in four dimensions, putting the component fields in correspondence with Pascal's pyramid at layer \mathcal{N} . This correspondence is a generic property of the \mathcal{N} -extended chiral sector.

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1 Introduction

Four-dimensional conformal field theories have attracted much attention in the last decade, mainly because of their relevance in the context of the AdS/CFT correspondence and its developments as well as its applications in completely different fields of which condensed matter is an example. The conformal symmetry imposes stringent constraints on the theory [1]. In the case of two dimensions, for example, where scale invariance implies conformal invariance, the symmetry is enough to solve the theory [2, 3]. In four and higher dimensions, this is in general not true [4], since examples are known of theories with scale but not conformal symmetry [5, 6], but meaningful statements can be made as well (e.g. [7, 8] for the four-dimensional case). Moreover, in the presence of supersymmetry, superconformal invariance puts restrictions on the operator scaling dimensions, independently of the space-time dimensionality [9, 10].

For these theories, a great deal of information can be obtained from the conformal group, that in four dimensions is $SO(4, 2)$ and that acts non-linearly on the coordinates, due to the presence of operations such as the inversion and the special conformal transformations. However, it has been long known that it is possible to formulate the theory in such a way that the conformal group will act linearly on the coordinates, in the same fashion as the angular momentum does. In addition, correlation functions automatically exhibit manifest conformal symmetry and any results written in terms of the embedding coordinates are valid not just in Minkowski space but in any conformally flat space-time. This approach goes back to [11] and had applications to M/string theory branes [12] as well as to more complicated conformal field theories in [13, 14] (in particular [14] generalizes the construction from scalars to more

generic tensor fields). Moreover, embeddings of chiral conformal superspaces were considered in [15], in terms of off-shell twistors [16], while [17] and [18] review the general (chiral and non-chiral) case. In particular, using the twistor notation, [18] gives the two-point correlators between chiral and anti-chiral superfields for arbitrary \mathcal{N} . In some cases, n -point correlators have been calculated for various values of n (see for example [19] where $\mathcal{N} = 4$ and the primaries are not chiral superfields). The supertwistor approach was also used in [20] to study the superconformal structure of 4d $\mathcal{N} = 2$ compactified harmonic/projective superspace. The results presented here (as well as in [21]) are equivalent to those obtained with twistor calculations¹.

In the embedding method, one introduces two additional coordinates and embeds the four-dimensional space-time into a larger six-dimensional space with signature $(-, -, +, +, +, +)$. In this space the conformal group generators act as angular momenta. The four-dimensional space is recovered by constraining the six-dimensional coordinates on the projective light-cone. The group $SO(4, 2)$ is isomorphic to $SU(2, 2)$. As a consequence, one can re-express everything in spinor indices and introduce gamma matrices for the basis change from $SO(4, 2)$ to $SU(2, 2)$.

In [21], the authors generalize the embedding methods to $\mathcal{N} = 1$ superconformal field theories. The superconformal group is $SU(2, 2|1)$ and the six-dimensional complex superspace is constructed out of the coordinates (X_{AB}, \bar{X}^{AB}) , which contain the standard space-time $X_{\alpha\beta}$, one fermionic direction θ_α and an additional bosonic coordinate φ . X_{AB} transforms linearly under a superconformal transformation. Upon projecting on the projective light-cone, the four-dimensional superspace $(x^\mu, \theta, \bar{\theta})$ is recovered. Moreover, scalar and holomorphic fields are considered and it is shown that they correspond to four-dimensional $\mathcal{N} = 1$ chiral superfields whose $\theta = \bar{\theta} = 0$ component is an $\mathcal{N} = 1$ chiral primary operator [21, 22].

In this paper we generalize the embedding method to extended supersymmetry where the superconformal group is $SU(2, 2|\mathcal{N})$. Most of the arguments are similar to the ones in [21] and analogue results are obtained by considering more fermionic coordinates $\theta_{I\alpha}$, with $I = 1, \dots, \mathcal{N}$. Automatically, more bosonic variables φ_{IJ} also appear in the six-dimensional superspace, which will then be removed once the light-cone constraint is imposed. We define such a set-up in Section 2, where the generators as well as the necessary representations are introduced. In Section 3 we construct the \mathcal{N} -extended four-dimensional superspace. This is done as a coset construction, achieved by applying translations and supersymmetry transformations to a reference origin. The resulting space is invariant under the superconformal and Lorentz groups. In Section 4 we consider representations of the superconformal algebra in terms of superfields. In particular, we generalize the arguments of [21] to \mathcal{N} -extended chiral superfields, find their transformation rules under the $U(\mathcal{N})_R$ symmetry subgroup of the superconformal algebra as well as under generic superconformal transformations. We also study in detail the case of chiral superfields in $\mathcal{N} = 2$ extended supersymmetry.

As an aside result of our calculations, we have discovered an interesting connection between the number of component fields of a given kind in an \mathcal{N} -extended supermultiplet and the so-called Pascal's pyramid at layer \mathcal{N} . This aspect of number theory seems not to be mentioned explicitly in the literature. One can use it as a check that superfield expansions are correct.

Throughout this paper we use the conventions of [21] and [23]. In particular, the various definitions for the spinor indices and the gamma matrices are the same as in [21]. In the appendix we give some details of the calculations as well as our notation. In particular, in Appendix A we compute explicitly the $SU(2, 2)$ transformations of the coordinates $X_{\alpha\beta}$. In Appendix B we give our notation for the spinorial indices and show some identities that will be useful in our computations. In Appendix C we provide the necessary for computing the coordinates of the 4d barred superspace. In Appendix D we define the Pascal pyramid, with some of its properties and symmetries, construct the $\mathcal{N} = 4$ chiral multiplet and relate it to the pyramid at layer 4.

¹We thank Warren Siegel for this remark.

2 Conformal and superconformal group

The superconformal group in four dimensions has fifteen generators that satisfy the $SO(4, 2)$ algebra. It includes the Poincaré generators (P^μ for translations and $M^{\mu\nu}$ for Lorentz transformations) as a subalgebra. In addition there are the generators K^μ for special conformal transformations as well as the dilatation generator D . Special conformal transformations can be thought of as an inversion, followed by a translation and then by another inversion and act non-linearly on the coordinates x^μ . Schematically:

$$P^\mu \quad x^\mu \rightarrow x^\mu + a^\mu \quad \delta x^\mu = a^\mu \quad (2.1a)$$

$$M^{\mu\nu} \quad x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu \quad (\Lambda \in SO(3, 1)) \quad \delta x^\mu = \omega^\mu{}_\nu x^\nu \quad (\omega_{\mu\nu} = -\omega_{\nu\mu}) \quad (2.1b)$$

$$K^\mu \quad x^\mu \rightarrow \frac{x^\mu + x^2 b^\mu}{1 + 2b \cdot x + b^2 x^2} \quad \delta x^\mu = x^2 b^\mu - 2(b \cdot x)x^\mu \quad (2.1c)$$

$$D \quad x^\mu \rightarrow \lambda x^\mu \quad \delta x^\mu = \epsilon x^\mu \quad (2.1d)$$

with real parameters a^μ , b^μ , $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ and $\lambda = 1 + \epsilon$. Here indices are raised and lowered by the four-dimensional metric $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.

The conformal group $SO(4, 2)$ is identical to the Lorentz group in a space with six dimensions and signature $(-, -, +, +, +, +)$. One can use this observation to make the conformal transformations act linearly on the coordinates. Define new variables X^m , with $m = +, \mu, -$. Under $SO(4, 2)$, X^m transforms linearly:

$$X^m \rightarrow \Lambda^m{}_n X^n, \quad \text{with} \quad \Lambda^T \eta \Lambda = \eta \quad \text{and} \quad \eta_{mn} = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & \eta_{\mu\nu} & 0 \\ 1/2 & 0 & 0 \end{pmatrix} \quad (2.2)$$

Infinitesimally, $\Lambda^m{}_n = \delta^m{}_n + \omega^m{}_n$, the conformal transformation $\delta X^m = \omega^{mn} X_n = \frac{i}{2} \omega^{pq} L_{pq} X^m$ is generated by the $SO(4, 2)$ differential operators

$$L^{mn} = i X^m \frac{\partial}{\partial X_n} - i X^n \frac{\partial}{\partial X_m}. \quad (2.3)$$

In order to recover the four-dimensional space, one has to demand that the coordinates X^m are constrained on the projective light-cone:

$$X^2 \equiv X^m \eta_{mn} X^n = 0, \quad X^m \sim \lambda X^m \quad (\lambda \in \mathbb{R}). \quad (2.4)$$

Since the conformal group $SO(4, 2)$ is equivalent to $SU(2, 2)$, we can transform everything into spinor notation. $SU(2, 2)$ is the group of four by four special matrices that are unitary with respect to a fixed invariant matrix of signature $\text{diag}(+, +, -, -)$. The connection is realized by using gamma matrices to transform the vector index m into an anti-symmetric pair of spinor indices (α, β) . A spinor V_α of $SU(2, 2)$ transforms as

$$V_\alpha \rightarrow U_\alpha{}^\beta V_\beta, \quad U \in SU(2, 2). \quad (2.5)$$

The $SU(2, 2)$ index $\alpha = 1, \dots, 4$ has an undotted component transforming in the fundamental representation of $SL(2, \mathbb{C})$ and a dotted component transforming in the complex conjugate representation, $\alpha = (a, \dot{a})$, with $a = 1, 2$ and $\dot{a} = 1, 2$ (e.g. [23, 24]). The vector X^m becomes an anti-symmetric tensor $X_{\alpha\beta} = -X_{\beta\alpha}$, with each index transforming as in (2.5):

$$X^m = \frac{1}{2} X_{\alpha\beta} \Gamma^{m\alpha\beta}, \quad X_{\alpha\beta} = \frac{1}{2} X_m \tilde{\Gamma}_{\alpha\beta}^m, \quad (2.6)$$

with

$$\tilde{\Gamma}_{\alpha\beta}^m = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} (\Gamma^m)^{\gamma\delta}. \quad (2.7)$$

The light-cone constraint is written as

$$X_{\alpha\beta}X^{\alpha\beta} = 0 \quad (2.8)$$

with $X^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}X_{\gamma\delta}$. Similarly, one can express the $SO(4, 2)$ generators L^{mn} in spinor notation. The result is $L_{\alpha}{}^{\beta} = -\frac{1}{2}(\Sigma^{mn})_{\alpha}{}^{\beta}L_{mn}$, $L^{mn} = -(\Sigma^{mn})_{\alpha}{}^{\beta}L_{\beta}{}^{\alpha}$, with $(\Sigma^{mn})_{\alpha}{}^{\beta} = -\frac{i}{4}\left(\tilde{\Gamma}^m\Gamma^n - \tilde{\Gamma}^n\Gamma^m\right)_{\alpha}{}^{\beta}$ being the $SU(2, 2)$ generators.

2.1 Superconformal algebra

The \mathcal{N} -extended superconformal group in four dimensions is $SU(2, 2|\mathcal{N})$ consisting of matrices of the form

$$\mathcal{U}_A{}^B = \begin{pmatrix} U_{\alpha}{}^{\beta} & \varphi_{\alpha}^J \\ \varkappa_I^{\beta} & z_I^J \end{pmatrix}. \quad (2.9)$$

This is a block matrix, with the indices running in the range $\alpha, \beta = 1, \dots, 4$ and $I, J = 1, \dots, \mathcal{N}$. The diagonal blocks are commuting, while the off-diagonal ones are anti-commuting. An $SU(2, 2|\mathcal{N})$ vector in the fundamental representation is written as

$$V_A = \begin{pmatrix} V_{\alpha} \\ \psi_I \end{pmatrix}, \quad (2.10)$$

where the V_{α} component is fermionic and the \mathcal{N} ψ_I fields are bosonic. Indices are raised and lowered with the invariant matrix

$$A^{\dot{A}B} = \begin{pmatrix} A^{\dot{\alpha}\beta} & 0 \\ 0 & \delta_I^J \end{pmatrix}, \quad (2.11)$$

where $A^{\dot{\alpha}\beta}$ is the $SU(2, 2)$ invariant metric. In order to be an element of $SU(2, 2|\mathcal{N})$, the matrix $\mathcal{U}_A{}^B$ must have unit superdeterminant:

$$(S \det U)^{-1} = (\det z)^{-1} \cdot \det(U - \varphi \cdot z^{-1} \cdot \varkappa) = 1. \quad (2.12)$$

It must also satisfy

$$\mathcal{U}_{\dot{B}}{}^{\dot{A}} A^{\dot{B}B} \mathcal{U}_B{}^A = A^{\dot{A}A}, \quad (2.13)$$

where $\mathcal{U}_{\dot{B}}{}^{\dot{A}} = (\mathcal{U}_A{}^B)^{\dagger}$.

One can introduce the $SU(2, 2|\mathcal{N})$ generators $\mathcal{T}_A{}^B$ as

$$\mathcal{U}_A{}^B = \delta_A{}^B + i\mathcal{T}_A{}^B. \quad (2.14)$$

Hence:

$$\mathcal{T}_A{}^B = \begin{pmatrix} T_{\alpha}{}^{\beta} & \phi_{\alpha}^J \\ \psi_I^{\beta} & \phi_I^J \end{pmatrix}, \quad (2.15)$$

with $U_{\alpha}{}^{\beta} = \delta_{\alpha}{}^{\beta} + iT_{\alpha}{}^{\beta}$, $z_I^J = \delta_I^J + i\phi_I^J$, $\phi_{\alpha}^J = -i\varphi_{\alpha}^J$ and $\psi_I^{\beta} = -i\varkappa_I^{\beta}$. From the superdeterminant, it follows that $\mathcal{T}_A{}^B$ have vanishing supertrace:

$$Str \mathcal{T} = T_{\alpha}{}^{\alpha} - \phi = 0, \quad (2.16)$$

where ϕ is the trace of ϕ_I^J , $\phi \equiv \sum_I \phi_I^I$. From (2.13), it follows that

$$A^{\dot{A}B} \mathcal{T}_B{}^A - \mathcal{T}_{\dot{B}}{}^{\dot{A}} A^{\dot{B}A} = 0. \quad (2.17)$$

Hence, explicitly

- for $(A, B) = (\alpha, \beta)$: $A^{\dot{\alpha}\beta}T_{\beta}^{\alpha} - T_{\dot{\beta}}^{\dot{\alpha}}A^{\dot{\beta}\alpha} = 0$, which implies that T_{α}^{β} is a $U(2, 2)$ generator
- for mixed indices: $\psi_I^{\alpha} = A^{\dot{\beta}\alpha}\phi_{\dot{\beta}}^I \equiv \bar{\phi}_I^{\alpha}$ (equivalently $\psi_I = \bar{\phi}_I$)
- for $(A, B) = (I, J)$: $\phi_I^J = (\phi^{\dagger})_I^J$ (equivalently $\phi = \phi^{\dagger}$), which states that ϕ_I^J is hermitian and hence $z_I^J = (e^{i\phi})_I^J$ is unitary, $z_I^J \in U(\mathcal{N})$, as it should be since ϕ_I^J is the R -symmetry generator.

In order to make T_{α}^{β} an $SU(2, 2)$ generator (and hence $U_{\alpha}^{\beta} \in SU(2, 2)$), one can subtract its trace and replace \mathcal{T}_A^B by

$$\mathcal{T}_A^B = \begin{pmatrix} T_{\alpha}^{\beta} + \frac{1}{4}\delta_{\alpha}^{\beta}\phi & \phi_{\alpha}^J \\ \bar{\phi}_I^{\beta} & \phi_I^J \end{pmatrix} \quad (2.18)$$

in the fundamental representation.

By defining supercharges according to

$$-i\delta V_A = [\bar{\phi}_J^{\beta}Q_{\beta}^J + \phi_{\alpha}^J\bar{Q}_J^{\beta}]V_A, \quad (2.19)$$

one has

$$(Q_{\sigma}^L)_A^B = \begin{pmatrix} 0 & 0 \\ \delta_{\sigma}^{\beta}\delta_I^L & 0 \end{pmatrix} \quad \text{and} \quad (\bar{Q}_L^{\sigma})_A^B = \begin{pmatrix} 0 & \delta_{\alpha}^{\sigma}\delta_L^J \\ 0 & 0 \end{pmatrix}. \quad (2.20)$$

Similarly, the R -symmetry generator is a non-abelian matrix

$$R_A^B = \begin{pmatrix} \frac{1}{4}\delta_{\alpha}^{\beta}\phi & 0 \\ 0 & \phi_I^J \end{pmatrix}, \quad (2.21)$$

which contains the $U(\mathcal{N})$ block of the extended supersymmetry.

2.2 Representations and invariants

Many concepts here are similar to those presented in [21]. If V_A transforms in the fundamental representation,

$$V_A = \begin{pmatrix} V_{\alpha} \\ \psi_I \end{pmatrix}, \quad V_A \rightarrow \mathcal{U}_A^B V_B \quad (2.22)$$

and $\bar{W}^A \equiv (W^{\dagger})_{\dot{B}}A^{\dot{B}A}$ in the anti-fundamental,

$$\bar{W}^A = \begin{pmatrix} \bar{W}^{\alpha} \\ \bar{\psi}^I \end{pmatrix}, \quad \bar{W}^A \rightarrow \bar{W}^B (\mathcal{U}^{-1})_B^A, \quad (2.23)$$

then the product $Z_A^B = V_A \bar{W}^B$ transforms as a tensor

$$Z_A^B \rightarrow \mathcal{U}_A^C Z_C^D (\mathcal{U}^{-1})_D^B. \quad (2.24)$$

The singlet representation is given by its supertrace,

$$\text{Str} Z_A^B = -Z_A^B \lambda_B^A, \quad (2.25)$$

where

$$\lambda_A^B = \begin{pmatrix} -\delta_{\alpha}^{\beta} & 0 \\ 0 & \delta_I^J \end{pmatrix}, \quad (2.26)$$

which, due to the property of the supertrace $\text{Str}(\mathcal{U}_1 \mathcal{U}_2) = \text{Str}(\mathcal{U}_2 \mathcal{U}_1)$, is invariant under $SU(2, 2|\mathcal{N})$ transformations. The adjoint representation is given by the part of Z_A^B with vanishing supertrace,

$$Z_A^B + \frac{1}{4 - \mathcal{N}} (Z_C^D \lambda_D^C) \delta_A^B. \quad (2.27)$$

Tensors X_{AB} and \bar{X}^{AB} transform² as the products $X_{AB} \sim V_A V_B$ and $\bar{X}^{AB} \sim \bar{V}^A \bar{V}^B$:

$$\begin{aligned} X_{AB} &\rightarrow \mathcal{U}_A{}^C X_{CD} \hat{\mathcal{U}}_B{}^D, \quad \hat{\mathcal{U}}_B{}^D = \begin{pmatrix} \mathcal{U}_\alpha{}^\beta & \mathcal{U}_\alpha{}^J \\ -\mathcal{U}_I{}^\beta & \mathcal{U}_I{}^J \end{pmatrix} \\ \bar{X}^{AB} &\rightarrow (\hat{\mathcal{U}}^{-1})_C{}^A \bar{X}^{CD} (\mathcal{U}^{-1})_D{}^B, \quad (\hat{\mathcal{U}}^{-1})_D{}^B = \begin{pmatrix} (\mathcal{U}^{-1})_\alpha{}^\beta & (\mathcal{U}^{-1})_\alpha{}^J \\ -(\mathcal{U}^{-1})_I{}^\beta & (\mathcal{U}^{-1})_I{}^J \end{pmatrix}. \end{aligned} \quad (2.28)$$

This implies that the product $X_{AC} \lambda_D{}^C \bar{X}^{DB}$ transforms as $Z_A{}^B$ and hence the scalar product

$$X \cdot \bar{Y} \equiv \text{Str}(X \lambda \bar{Y}) = \text{Tr}(-\lambda X \lambda \bar{Y}) \quad (2.29)$$

is $SU(2, 2|\mathcal{N})$ invariant.

3 Superspace

The tensor X_{AB} contains the superspace components of six-dimensional conformal group. Explicitly:

$$X_{AB} = \begin{pmatrix} X_{\alpha\beta} & \theta_{J\alpha} \\ \theta_{I\beta} & \varphi_{IJ} \end{pmatrix}, \quad (3.1)$$

where $X_{\alpha\beta}$ is anti-symmetric, $\theta_{I\alpha} \equiv X_{I\alpha} = X_{\alpha I}$ are fermionic variables and $\varphi_{IJ} \equiv X_{IJ}$ is a symmetric matrix of bosonic coordinates. In order to use the invariant scalar product (2.29), we need the conjugate of X , \bar{X}^{AB} , with components

$$\bar{X}^{AB} = \begin{pmatrix} \bar{X}^{\alpha\beta} & \bar{\theta}^{J\alpha} \\ \bar{\theta}^{I\beta} & \bar{\varphi}^{IJ} \end{pmatrix}. \quad (3.2)$$

Defining $X_{\dot{\alpha}\dot{\beta}} = (X_{\alpha\beta})^\dagger$ and $\theta_{\dot{\alpha}}^I = (\theta_{I\alpha})^\dagger$, and using the invariant matrix (2.11), the components are $\bar{X}^{\alpha\beta} = A^{\dot{\alpha}\alpha} A^{\dot{\beta}\beta} X_{\dot{\alpha}\dot{\beta}}$, $\bar{\theta}^{I\alpha} = A^{\dot{\alpha}\alpha} \theta_{\dot{\alpha}}^I$, $\bar{\varphi}^{IJ} = (\varphi_{IJ})^\dagger$. The full superspace is then described by the supercoordinates (X_{AB}, \bar{X}^{AB}) , endowed with the scalar product (2.29). In components:

$$X_1 \cdot \bar{X}_2 = X_{1\alpha\beta} \bar{X}_2^{\alpha\beta} + 2\theta_{1I\alpha} \bar{\theta}_2^{I\alpha} - \varphi_{1IJ} \bar{\varphi}_2^{IJ}. \quad (3.3)$$

The superconformal transformations of the coordinates can be computed by using (2.28). For X_{AB} , they are $-i\delta X_{AB} = X_{AC} \hat{\mathcal{T}}_B{}^C + \mathcal{T}_A{}^C X_{CB}$, or in components:

$$-i\delta X_{\alpha\beta} = T_\alpha{}^\gamma X_{\gamma\beta} + T_\beta{}^\gamma X_{\alpha\gamma} + \phi_\alpha^I \theta_{I\beta} - \phi_\beta^I \theta_{I\alpha} + \frac{1}{2} \phi X_{\alpha\beta} \quad (3.4a)$$

$$-i\delta \theta_{I\alpha} = T_\alpha{}^\gamma \theta_{I\gamma} + \bar{\phi}_I{}^\beta X_{\beta\alpha} + \phi_\alpha^J \varphi_{JI} + \phi_I^J \theta_{J\alpha} + \frac{1}{4} \phi \theta_{I\alpha} \quad (3.4b)$$

$$-i\delta \varphi_{IJ} = \bar{\phi}_J{}^\gamma \theta_{I\gamma} + \bar{\phi}_I{}^\gamma \theta_{J\gamma} + \phi_J^K \varphi_{IK} + \phi_I^K \varphi_{KJ} \quad (3.4c)$$

For \bar{X}^{AB} , they are $i\delta \bar{X}^{AB} = \bar{X}^{AC} \mathcal{T}_C{}^B + \hat{\mathcal{T}}_C{}^A \bar{X}^{CB}$, or in components:

$$i\delta \bar{X}^{\alpha\beta} = T_\gamma{}^\alpha \bar{X}^{\gamma\beta} + T_\gamma{}^\beta \bar{X}^{\alpha\gamma} + \bar{\phi}_I{}^\alpha \bar{\theta}^{I\beta} - \bar{\phi}_I{}^\beta \bar{\theta}^{I\alpha} + \frac{1}{2} \phi \bar{X}^{\alpha\beta} \quad (3.5a)$$

$$i\delta \bar{\theta}^{I\alpha} = T_\gamma{}^\alpha \bar{\theta}^{I\gamma} - \phi_\gamma^I \bar{X}^{\gamma\alpha} + \bar{\phi}_J{}^\alpha \bar{\varphi}^{IJ} + \phi_J^I \bar{\theta}^{J\alpha} + \frac{1}{4} \phi \bar{\theta}^{I\alpha} \quad (3.5b)$$

$$i\delta \bar{\varphi}^{IJ} = -\phi_\gamma^I \bar{\theta}^{J\gamma} - \phi_\gamma^J \bar{\theta}^{I\gamma} + \phi_K^I \bar{\varphi}^{KJ} + \phi_K^J \bar{\varphi}^{IK} \quad (3.5c)$$

²To check these transformations, use the inversion formula for block matrices. One convenient expression is:

$$(\mathcal{U}^{-1})_A{}^B = \begin{pmatrix} (U^{-1} + U^{-1} \phi (z - \psi U^{-1} \phi)^{-1} \psi U^{-1})_\alpha{}^\beta & -(U^{-1} \phi (z - \psi U^{-1} \phi)^{-1})_\alpha{}^J \\ -((z - \psi U^{-1} \phi)^{-1} \psi U^{-1})_I{}^\beta & ((z - \psi U^{-1} \phi)^{-1})_I{}^J \end{pmatrix}.$$

3.1 4D superspace

In this section we construct the four-dimensional superspace, described by the projective coordinates $(X_{AB}, \bar{X}^{AB}) \sim (\lambda X_{AB}, \bar{\lambda} \bar{X}^{AB})$, $\lambda \in \mathbb{C}$, on the six-dimensional light-cone

$$X \cdot \bar{X} = 0. \quad (3.6)$$

We start from the origin defined as

$$\hat{X}_{\alpha\beta} = \frac{1}{2} \begin{pmatrix} i\epsilon_{ab}X^+ & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{\theta}_{I\alpha} = 0, \quad \hat{\varphi}_{IJ} = 0. \quad (3.7)$$

One can check that the origin satisfies the light-cone constraint³. An arbitrary point in superspace is reached by applying all possible superconformal transformations (3.4) and (3.5). Under superconformal transformations, the origin transforms as

$$-i\delta\hat{X}_{\alpha\beta} = \frac{1}{2}iX^+ \left(\delta_\beta^b T_\alpha^c \epsilon_{cb} + \delta_\alpha^a T_\beta^c \epsilon_{ac} + \delta_\alpha^a \delta_\beta^b \frac{1}{2} \epsilon_{ab} \phi \right) \quad (3.9a)$$

$$-i\delta\hat{\theta}_{I\alpha} = -\delta_\alpha^a \frac{1}{2} i\epsilon_{ab} \bar{\phi}_I^b X^+ \quad (3.9b)$$

$$-i\delta\hat{\varphi}_{IJ} = 0. \quad (3.9c)$$

Explicitly, from table 1 in Appendix A, the origin is invariant under special conformal and Lorentz transformations, and projectively invariant under dilatations. It is also invariant under supersymmetry transformations generated by a spinor of the form

$$\phi_\alpha^I \sim \begin{pmatrix} \eta_a^I \\ 0 \end{pmatrix}, \quad (3.10)$$

because in this case $\bar{\phi}_I^\beta$ in (3.9) has only the down (dotted) component. Hence a generic point in superspace is reached from the origin by applying translations with parameter $\omega^{\mu-} = 2\delta x^\mu = 2x^\mu$, and supersymmetry transformations with spinorial parameter

$$\phi_\alpha^I = \begin{pmatrix} 0 \\ 2\bar{\theta}^{I\dot{a}} \end{pmatrix}. \quad (3.11)$$

Then the four-dimensional superspace is a coset space where points are labelled by $(x^\mu, \{\theta_{Ia}\}, \{\bar{\theta}^{J\dot{a}}\})$ and the symmetries above have been modded out. We denote the whole set of theta coordinates by curly brackets:

$$\{\theta_{Ia}\} = \{\theta_{1a}, \theta_{2a}, \dots, \theta_{Na}\} \quad \text{and} \quad \{\bar{\theta}^{I\dot{a}}\} = \{\bar{\theta}^{1\dot{a}}, \bar{\theta}^{2\dot{a}}, \dots, \bar{\theta}^{N\dot{a}}\}. \quad (3.12)$$

The explicit transformation that brings the origin to the point $(x^\mu, \{\theta_{Ia}\}, \{\bar{\theta}^{J\dot{a}}\})$ in superspace is a product of two commuting transformations, one in ordinary space-time and the other one in the theta directions:

$$\mathcal{U}(x^\mu, \{\theta_{Ia}\}, \{\bar{\theta}^{J\dot{a}}\})_A^B = \mathcal{U}(x^\mu, 0, 0)_A^C \mathcal{U}(0, \{\theta_{Ia}\}, \{\bar{\theta}^{J\dot{a}}\})_C^B. \quad (3.13)$$

The space-time part is as in [21]:

$$\mathcal{U}(x^\mu, 0, 0)_A^B = \begin{pmatrix} \left(e^{-ix_\mu \Sigma^{\mu+}} \right)_\alpha^\beta & 0 \\ 0 & \delta_I^J \end{pmatrix} = \begin{pmatrix} \delta_a^b & 0 & 0 \\ ix^\mu (\bar{\sigma}_\mu)^{\dot{a}b} & \delta_{\dot{a}}^{\dot{b}} & 0 \\ 0 & 0 & \delta_I^J \end{pmatrix}, \quad (3.14)$$

³Since:

$$\hat{X}^{\alpha\beta} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & i\epsilon_{\dot{a}\dot{b}} \bar{X}^+ \end{pmatrix}. \quad (3.8)$$

where δ_I^J is the $\mathcal{N} \times \mathcal{N}$ identity matrix. The theta part is a generalization of the expression used in [21] (see Appendix B):

$$\mathcal{U}(0, \{\theta_{Ia}\}, \{\bar{\theta}^{J\dot{a}}\})_A^B = \begin{pmatrix} \delta_\alpha^\beta - \frac{1}{2} \sum_I \phi_\alpha^I \bar{\phi}_I^\beta & i\phi_\alpha^J \\ i\bar{\phi}_I^\beta & \delta_I^J \end{pmatrix} = \begin{pmatrix} -\sum_I \theta_I \sigma^\mu \bar{\theta}^I (\bar{\sigma}_\mu)^{\dot{a}b} & 0 & 0 \\ 2i\theta_I^b & \delta_{\dot{a}b} & 2i\bar{\theta}^{J\dot{a}} \\ 0 & 0 & \delta_I^J \end{pmatrix}. \quad (3.15)$$

This matrix is also equal to the product of analogous matrices, but with only one theta coordinate different from zero:

$$\mathcal{U}(0, \{\theta_{Ia}\}, \{\bar{\theta}^{J\dot{a}}\}) = \prod_{I=1}^{\mathcal{N}} \mathcal{U}(0, \theta_{Ia}, \bar{\theta}^{I\dot{a}}) = \mathcal{U}(0, \theta_{1a}, \bar{\theta}^{1\dot{a}}) \mathcal{U}(0, \theta_{2a}, \bar{\theta}^{2\dot{a}}) \cdots \mathcal{U}(0, \theta_{\mathcal{N}a}, \bar{\theta}^{\mathcal{N}\dot{a}}), \quad (3.16)$$

where in the I -th factor $\theta_{Ja} = \bar{\theta}^{J\dot{a}} = 0$ if $J \neq I$. One can explicitly check that each of the single factors commutes with each other as well as with the space-time transformation (3.14) and that, consequently, the full theta matrix (3.15) commutes with (3.14) too. The complete transformation from the origin to any point in the superspace is explicitly

$$\mathcal{U}(x^\mu, \theta_{Ia}, \bar{\theta}^{J\dot{a}})_A^B = \begin{pmatrix} \delta_a^b & 0 & 0 \\ iy^\mu (\bar{\sigma}_\mu)^{\dot{a}b} & \delta_{\dot{a}b} & 2i\bar{\theta}^{J\dot{a}} \\ 2i\theta_I^b & 0 & \delta_I^J \end{pmatrix}, \quad (3.17)$$

where

$$y^\mu = x^\mu + i \sum_I \theta_I \sigma^\mu \bar{\theta}^I \quad (3.18)$$

is the coordinate of the \mathcal{N} -extended chiral superspace⁵.

We use eq. (3.17) to deduce the coordinates of the 4d superspace. Starting from the origin (3.7) and using (2.28), or equivalently (3.4)-(3.5), one has⁶:

$$X_{\alpha\beta}(y, \theta) = \mathcal{U}_\alpha^\gamma \hat{X}_{\gamma\delta} \mathcal{U}_\beta^\delta = \frac{i}{2} X^+ \begin{pmatrix} \epsilon_{ab} & \epsilon_{ad} iy^\mu (\bar{\sigma}_\mu)^{\dot{b}d} \\ iy^\mu (\bar{\sigma}_\mu)^{\dot{a}c} \epsilon_{cb} & -y^\mu y^\nu (\bar{\sigma}_\mu)^{\dot{a}c} (\bar{\sigma}_\nu)^{\dot{b}d} \epsilon_{cd} \end{pmatrix} \equiv \frac{1}{2} Y_m \tilde{\Gamma}_{\alpha\beta}^m \quad (3.19a)$$

$$\theta_{I\alpha}(y, \theta) = -\mathcal{U}_\alpha^\gamma \hat{X}_{\gamma\delta} \mathcal{U}_I^\delta = X^+ \begin{pmatrix} \theta_{Ia} \\ iy^\mu (\bar{\sigma}_\mu)^{\dot{a}b} \theta_{Ib} \end{pmatrix} \quad (3.19b)$$

$$\varphi_{IJ}(y, \theta) = -\mathcal{U}_I^\gamma \hat{X}_{\gamma\delta} \mathcal{U}_J^\delta = 2i X^+ \theta_I \cdot \theta_J. \quad (3.19c)$$

where

$$Y^m = (Y^+ = X^+, Y^\mu = X^+ y^\mu, Y^- = -X^+ y^2) \quad (3.20)$$

and we use the epsilon tensor to lower spinorial indices, $\theta_{Ia} = \epsilon_{ab} \theta_I^b$, so that $\theta_I \cdot \theta_J = \theta_I^c \epsilon_{cd} \theta_J^d = \theta_J \cdot \theta_I$.

⁴ E.g. for $\mathcal{N} = 2$ one has:

$$\begin{pmatrix} \delta_a^b & 0 & 0 & 0 \\ -\sum_I \theta_I \sigma^\mu \bar{\theta}^I (\bar{\sigma}_\mu)^{\dot{a}b} & \delta_{\dot{a}b} & 2i\bar{\theta}^{1\dot{a}} & 2i\bar{\theta}^{2\dot{a}} \\ 2i\theta_1^b & 0 & 1 & 0 \\ 2i\theta_2^b & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \delta_a^b & 0 & 0 & 0 \\ -\theta_1 \sigma^\mu \bar{\theta}^1 (\bar{\sigma}_\mu)^{\dot{a}b} & \delta_{\dot{a}b} & 2i\bar{\theta}^{1\dot{a}} & 0 \\ 2i\theta_1^b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \delta_a^b & 0 & 0 & 0 \\ -\theta_2 \sigma^\mu \bar{\theta}^2 (\bar{\sigma}_\mu)^{\dot{a}b} & \delta_{\dot{a}b} & 0 & 2i\bar{\theta}^{2\dot{a}} \\ 0 & 0 & 1 & 0 \\ 2i\theta_2^b & 0 & 0 & 1 \end{pmatrix}.$$

⁵See Section 3 (formula (3.3.28) in particular) of [24] for any \mathcal{N} , or e.g. [25] explicitly for $\mathcal{N} = 2$.

⁶See Appendix B.

Similarly, for the barred coordinates one finds (more details in Appendix C):

$$\bar{X}^{\alpha\beta} = (\mathcal{U}^{-1})_{\dot{c}}^{\alpha} \hat{X}^{\dot{c}\dot{d}} (\mathcal{U}^{-1})_{\dot{d}}^{\beta} = \frac{1}{2} \bar{Y}_m \Gamma^{m\alpha\beta} \quad (3.21a)$$

$$\bar{\theta}^{I\alpha} = (\mathcal{U}^{-1})_{\dot{c}}^I \hat{X}^{\dot{c}\dot{d}} (\mathcal{U}^{-1})_{\dot{d}}^{\alpha} = \bar{X}^+ \left(\frac{-i\bar{y}^{\mu} (\bar{\sigma}_{\mu})^{ba} \bar{\theta}_b^I}{\bar{\theta}_a^I} \right) \quad (3.21b)$$

$$\bar{\varphi}^{IJ} = (\mathcal{U}^{-1})_{\dot{c}}^I \hat{X}^{\dot{c}\dot{d}} (\mathcal{U}^{-1})_{\dot{d}}^J = -2i \bar{X}^+ \bar{\theta}^I \cdot \bar{\theta}^J, \quad (3.21c)$$

where

$$\bar{Y}^m = (\bar{Y}^+ = \bar{X}^+, \bar{Y}^{\mu} = \bar{X}^+ \bar{y}^{\mu}, Y^- = -\bar{X}^+ \bar{y}^2) = (Y^m)^{\dagger}, \quad (3.22)$$

$\bar{y}^{\mu} = x^{\mu} - i \sum_I \theta_I \sigma^{\mu} \bar{\theta}^I$ and we have lowered the theta index, $\bar{\theta}_a^I = \epsilon_{ab} \bar{\theta}^{Ib}$.

Under superconformal transformations, (y^{μ}, θ_a) transform as the coordinates of the \mathcal{N} -extended chiral superspace. In fact, using (3.4) with

$$\phi_{\alpha}^I = \left(\frac{-2\eta_a^I}{2\bar{\epsilon}^{I\dot{a}}} \right) \Longleftrightarrow \bar{\phi}_I^{\alpha} = (2\epsilon_I^a, -2\bar{\eta}_{I\dot{a}}) \quad (3.23)$$

as the only non-vanishing parameter, we get the following transformations:

$$\delta y^{\mu} = \sum_I (2i\theta_I \sigma^{\mu} \bar{\epsilon}^I - 2y_{\nu} \theta_I \sigma^{\mu} \bar{\sigma}^{\nu} \eta^I), \quad (3.24a)$$

$$\delta \theta_I^a = \epsilon_I^a - i y^{\mu} (\bar{\sigma}_{\mu})^{ba} \bar{\eta}_b^I + 4 \sum_J \theta_I \cdot \theta_J \eta^{Ja}. \quad (3.24b)$$

Including also the barred half superspace given by \bar{X}^{AB} , one gets the full superspace spanned by the coordinates $(x = (y + \bar{y})/2, \theta_I, \bar{\theta}_I)$.

4 The chiral sector

$\mathcal{N} = 1$ conformal chiral superfields, $\Phi(y^{\mu}, \theta_a)$, were considered in [21]. Defining a chiral superfield on the light-cone

$$\Phi(y^{\mu}, \theta_a) \equiv (X^+)^{\Delta} \Phi(X_{AB}), \quad (4.1)$$

and expanding in powers of theta

$$\Phi(y^{\mu}, \theta_a) = A(y) + \sqrt{2} \theta \psi(y) + \theta^2 F(y), \quad (4.2)$$

with $y^{\mu} = x^{\mu} + i\theta \sigma^{\mu} \bar{\theta}$, it was checked that the component fields transform as a chiral multiplet under the $\mathcal{N} = 1$ super Poincaré group and have non-trivial rules under the special conformal transformations. In particular, the highest component $A(0)$ is a chiral primary operator, in the sense that it is annihilated by the special superconformal generators S_a and $\bar{S}^{\dot{a}}$, which are related to the supercharges by

$$Q_{\alpha} = \frac{i}{2} \begin{pmatrix} -Q_a \\ \bar{S}^{\dot{a}} \end{pmatrix} \quad \text{and} \quad \bar{Q}^{\alpha} = -\frac{i}{2} \begin{pmatrix} S^a \\ -\bar{Q}_{\dot{a}} \end{pmatrix}. \quad (4.3)$$

Moreover, under the $U(1)_R \subset SU(2, 2|1)$ R -symmetry, one recovers the fact that the R -charge is proportional to the scaling dimension Δ at a superconformal fixed point [21], according to

$$\Phi'(y, e^{3i\phi/4}\theta) = e^{i\frac{\phi}{2}\Delta} \Phi(y, \theta). \quad (4.4)$$

Actually, this result continues to hold in the case of extended supersymmetry. Under $U(\mathcal{N})_R$ the indices I, J, \dots transform with a unitary matrix:

$$-i\delta X_{\alpha\beta} = \frac{1}{2}\phi X_{\alpha\beta} \rightarrow X'_{\alpha\beta} = e^{i\frac{\phi}{2}} X_{\alpha\beta} \quad (4.5a)$$

$$-i\delta\theta_{I\alpha} = \phi_I^J \theta_{J\alpha} + \frac{1}{4}\phi\theta_{I\alpha} \rightarrow \theta'_{I\alpha} = e^{i\frac{\phi}{4}} (e^{i\phi})_I^J \theta_{J\alpha} \quad (4.5b)$$

$$-i\delta\varphi_{IJ} = \phi_J^K \varphi_{IK} + \phi_I^K \varphi_{KJ} \rightarrow \varphi'_{IJ} = (e^{i\phi})_I^K (e^{i\phi})_J^L \varphi_{KL}. \quad (4.5c)$$

On the light-cone we define

$$\Phi(y, \theta_{Ia}) \equiv (X^+)^{\Delta} \Phi(X_{\alpha\beta}, \theta_{I\alpha}, \varphi_{IJ}), \quad (4.6)$$

where the components are functions of (y, θ_{Ia}) . Hence, as Φ is a scalar, i.e.

$$\Phi'(X'_{\alpha\beta}, \theta'_{I\alpha}, \varphi'_{IJ}) = \Phi(X_{\alpha\beta}, \theta_{I\alpha}, \varphi_{IJ}), \quad (4.7)$$

we find

$$\Phi'(y', \theta'_{Ia}) = \left(e^{i\frac{\phi}{2}} X^+\right)^{\Delta} \Phi'(X'_{\alpha\beta}, \theta'_{I\alpha}, \varphi'_{IJ}) = e^{i\frac{\phi}{2}\Delta} \Phi(y, \theta_{Ia}), \quad (4.8)$$

under $U(\mathcal{N})_R \subset SU(2, 2|\mathcal{N})$. This result is the same as in the $\mathcal{N} = 1$ case, as we wanted to show. In the above equation we have used that $X'^+ = e^{i\frac{\phi}{2}} X^+$ and hence $y' = y$, $\theta'_{Ia} = e^{i\frac{3}{4}\phi} (e^{i\phi})_I^J \theta_{Ja}$, as well as the projective property $\Phi(\lambda X_{AB}) = \lambda^{-\Delta} \Phi(X_{AB})$ with $\lambda = (X^+)^{-1}$.

Under a generic superconformal transformation generated by the spinor (3.23), using the scaling property of the superfield $\Phi(y, \theta_I)$ as well as its scalar nature, we have

$$\delta\Phi(y, \theta_I) = \Delta \frac{\delta X^+}{X^+} \Phi(y, \theta_I) - \delta y^\mu \frac{\partial \Phi}{\partial y^\mu}(y, \theta_I) - \delta\theta_I^a \frac{\partial \Phi}{\partial \theta_I^a}(y, \theta_I) \quad (4.9)$$

where $\delta X^+ = -4X^+ \theta_I \cdot \eta^I$ and the variations δy^μ and $\delta\theta_I^a$ are given by (3.24). The transformations for the component fields can be read off directly from this one by equating same powers of theta on both sides. Special superconformal transformations are obtained by setting $\epsilon_I^a = 0$ in these equations.

\mathcal{N} -extended superfields can be represented in terms of $\mathcal{N} = 1$ superfields by expanding all the theta coordinates but one. In the case of \mathcal{N} -extended *chiral* superfields, the component fields are $\mathcal{N} = 1$ *chiral* superfields and one can repeat the above argument. The four-dimensional expansion strongly depends on the value of \mathcal{N} , since the number of components grows exponentially with \mathcal{N} . In the following Subsection we will consider $\mathcal{N} = 2$.

4.1 $\mathcal{N} = 2$ chiral superfields

As an example, consider an $\mathcal{N} = 2$ chiral superfield, that we denote by $\Psi(x, \theta_1, \bar{\theta}^1, \theta_2, \bar{\theta}^2)$. The solution to the the chiral constraints $\bar{D}_{1\dot{a}}\Psi = 0$ and $\bar{D}_{2\dot{a}}\Psi = 0$ implies that Ψ does not depend explicitly on $\bar{\theta}^1$ and $\bar{\theta}^2$, but implicitly through the combination $y^\mu = x^\mu + i\theta_1\sigma^\mu\bar{\theta}^1 + i\theta_2\sigma^\mu\bar{\theta}^2$ (cf. (3.18 and [25]]):

$$\Psi \equiv \Psi(y, \theta_1, \theta_2). \quad (4.10)$$

Expanding in powers of, say, θ_2 , one finds (e.g. [25])

$$\Psi(y, \theta_1, \theta_2) = A(y, \theta_1) + \sqrt{2}\theta_2 W(y, \theta_1) + (\theta_2)^2 G(y, \theta_1), \quad (4.11)$$

where $A(y, \theta_1)$ and $G(y, \theta_1)$ are $\mathcal{N} = 1$ scalar superfields and $W^a(y, \theta_1)$ is an $\mathcal{N} = 1$ spinor superfield. However, for our purposes, we will consider the full expansion in θ_1 and θ_2 :

$$\begin{aligned} \Psi(X_{\alpha\beta}, \theta_{1\alpha}, \theta_{2\alpha}, \varphi_{11}, \varphi_{12}, \varphi_{22}) &= A + \sqrt{2}\theta_{1\alpha}\lambda^\alpha + \sqrt{2}\theta_{2\alpha}\chi^\alpha + \theta_{1\alpha}\theta_{2\beta}Z^{\alpha\beta} + \theta_{1\alpha}\theta_{1\beta}V^{\alpha\beta} + \\ &\quad \theta_{2\alpha}\theta_{2\beta}Y^{\alpha\beta} + \theta_{1\alpha}\theta_{1\beta}\theta_{2\gamma}E^{\alpha\beta\gamma} + \theta_{2\alpha}\theta_{2\beta}\theta_{1\gamma}F^{\alpha\beta\gamma} + \\ &\quad \theta_{1\alpha}\theta_{1\beta}\theta_{2\gamma}\theta_{2\delta}D^{\alpha\beta\gamma\delta} + \dots \end{aligned} \quad (4.12)$$

Here all the components are functions of $(X, \varphi) \equiv (X_{\alpha\beta}, \varphi_{11}, \varphi_{12}, \varphi_{22})$ and the dots represent higher-order terms (cubic and quartic in $\theta_{1\alpha}$ and $\theta_{2\alpha}$ as well as products thereof) that will vanish on the light-cone. The fields V , Y , E , F and D are anti-symmetric in $\alpha\beta$ (as well as in $\gamma\delta$ for D). All the components transform as follows under a scaling transformation of the coordinates: $A(\lambda X, \lambda\varphi) = \lambda^{-\Delta} A(X, \varphi)$, $\lambda^\alpha(\lambda X, \lambda\varphi) = \lambda^{-\Delta-1} \lambda^\alpha(X, \varphi)$ (same for χ^α), $Z^{\alpha\beta}(\lambda X, \lambda\varphi) = \lambda^{-\Delta-2} Z^{\alpha\beta}(X, \varphi)$ (same for $V^{\alpha\beta}$ and $Y^{\alpha\beta}$), etc. The four-dimensional $\mathcal{N} = 2$ chiral superfield is defined on the light-cone as

$$\begin{aligned} \Psi(y, \theta_{1a}, \theta_{2a}) \equiv & (X^+)^{\Delta} \Psi(X_{\alpha\beta}, \theta_{1\alpha}, \theta_{2\alpha}, \varphi_{11}, \varphi_{12}, \varphi_{22}) = \\ & A(y) + \sqrt{2}\theta_{1a}\lambda^a(y) + \sqrt{2}\theta_{2a}\chi^a(y) + \theta_{1a}\theta_{2b}Z^{ab}(y) + (\theta_1)^2 B(y) + \\ & (\theta_2)^2 C(y) + (\theta_1)^2 \theta_{2a} E^a(y) + (\theta_2)^2 \theta_{1a} F^a(y) + (\theta_1)^2 (\theta_2)^2 D(y). \end{aligned} \quad (4.13)$$

In the following, we will be using such a notation for the components of this $\mathcal{N} = 2$ off-shell multiplet. There are a total of $9 = 3^2$ fields, of which:

- 1 spin-0 scalar field A
- 2 spin- $\frac{1}{2}$ fields λ and χ
- 1 spin-1 tensor field Z
- 2 spin-1 fields B and C
- 2 spin- $\frac{3}{2}$ fields E and F
- 1 spin-2 field D .

This list agrees with the ones in e.g. [26, 27]. These numbers make up the Pascal pyramid⁷ at layer 2:

$$\begin{array}{ccccc} 1 & 2 & 1 & & \\ & 2 & 2 & & \\ & & 1 & & \end{array} \iff \begin{array}{ccccc} A & & \lambda, \chi & & Z \\ & B, C & & E, F & \\ & & D & & \end{array} \quad (4.14)$$

In Appendix D we consider the $\mathcal{N} = 4$ case: there, the triangle analogue to (4.14) is larger, contains 15 entries and corresponds to the pyramid at layer 4. More generically, the component fields of various kind in an \mathcal{N} -extended supermultiplets are equal in number to the elements of the Pascal pyramid at layer \mathcal{N} . The connection with the trinomial with power \mathcal{N} , which is at the origin of the pyramid, arises from the fact that in the product

$$\theta_{1\alpha_1}^{x_1} \theta_{2\alpha_2}^{x_2} \dots \theta_{\mathcal{N}\alpha_{\mathcal{N}}}^{x_{\mathcal{N}}} \quad (4.15)$$

the powers x_i can only have three values, $x_i = 0, 1, 2$, with $i = 1, \dots, \mathcal{N}$.

⁷In Appendix D we give all the necessary material about the Pascal pyramid. For the interested reader, more technical information can be found in [28, 29].

Using standard techniques, we can now identify the component fields. We find:

$$A(y) = (X^+)^\Delta A(X_{\alpha\beta}, 0) \quad (4.16a)$$

$$\lambda^a(y) = (X^+)^{\Delta+1} [\lambda^a(X_{\alpha\beta}, 0) - iy^\mu (\bar{\sigma}_\mu)^{\dot{a}a} \lambda_{\dot{a}}(X_{\alpha\beta}, 0)] \quad (4.16b)$$

$$\chi^a(y) = (X^+)^{\Delta+1} [\chi^a(X_{\alpha\beta}, 0) - iy^\mu (\bar{\sigma}_\mu)^{\dot{a}a} \chi_{\dot{a}}(X_{\alpha\beta}, 0)] \quad (4.16c)$$

$$Z^{ab}(y) = i(X^+)^{\Delta+1} \epsilon^{ab} \left[-4 \frac{\partial A}{\partial \varphi_{12}}(X_{\alpha\beta}, 0) + X_{\alpha\beta} Z_{\text{aver}}^{\alpha\beta}(X_{\alpha\beta}, 0) \right] \quad (4.16d)$$

$$B(y) = i(X^+)^{\Delta+1} \left[2 \frac{\partial A}{\partial \varphi_{11}}(X_{\alpha\beta}, 0) - X_{\alpha\beta} V^{\alpha\beta}(X_{\alpha\beta}, 0) \right] \quad (4.16e)$$

$$C(y) = i(X^+)^{\Delta+1} \left[2 \frac{\partial A}{\partial \varphi_{22}}(X_{\alpha\beta}, 0) - X_{\alpha\beta} Y^{\alpha\beta}(X_{\alpha\beta}, 0) \right] \quad (4.16f)$$

$$\begin{aligned} E^a(y) = i(X^+)^{\Delta+2} & \left[2\sqrt{2} \frac{\partial \lambda^a}{\partial \varphi_{12}}(X_{\alpha\beta}, 0) + 2\sqrt{2} \frac{\partial \chi^a}{\partial \varphi_{11}}(X_{\alpha\beta}, 0) \right. \\ & - 2\sqrt{2} iy^\mu (\bar{\sigma}_\mu)^{\dot{a}a} \frac{\partial \lambda_{\dot{a}}}{\partial \varphi_{12}}(X_{\alpha\beta}, 0) - 2\sqrt{2} iy^\mu (\bar{\sigma}_\mu)^{\dot{a}a} \frac{\partial \chi_{\dot{a}}}{\partial \varphi_{11}}(X_{\alpha\beta}, 0) \\ & \left. - X_{\alpha\beta} E^{\alpha\beta a}(X_{\alpha\beta}, 0) - X_{\alpha\beta} iy^\mu (\bar{\sigma}_\mu)^{\dot{a}a} E^{\alpha\beta}_{\dot{a}}(X_{\alpha\beta}, 0) \right] \end{aligned} \quad (4.16g)$$

$$\begin{aligned} F^a(y) = i(X^+)^{\Delta+2} & \left[2\sqrt{2} \frac{\partial \chi^a}{\partial \varphi_{12}}(X_{\alpha\beta}, 0) + 2\sqrt{2} \frac{\partial \lambda^a}{\partial \varphi_{22}}(X_{\alpha\beta}, 0) \right. \\ & - 2\sqrt{2} iy^\mu (\bar{\sigma}_\mu)^{\dot{a}a} \frac{\partial \chi_{\dot{a}}}{\partial \varphi_{12}}(X_{\alpha\beta}, 0) - 2\sqrt{2} iy^\mu (\bar{\sigma}_\mu)^{\dot{a}a} \frac{\partial \lambda_{\dot{a}}}{\partial \varphi_{11}}(X_{\alpha\beta}, 0) \\ & \left. - X_{\alpha\beta} F^{\alpha\beta a}(X_{\alpha\beta}, 0) - X_{\alpha\beta} iy^\mu (\bar{\sigma}_\mu)^{\dot{a}a} F^{\alpha\beta}_{\dot{a}}(X_{\alpha\beta}, 0) \right] \end{aligned} \quad (4.16h)$$

$$\begin{aligned} D(y) = (X^+)^{\Delta+2} & \left[2 \frac{\partial^2 A}{\partial \varphi_{12}^2}(X_{\alpha\beta}, 0) - 2X_{\alpha\beta} \frac{\partial Z_{\text{aver}}^{\alpha\beta}}{\partial \varphi_{12}}(X_{\alpha\beta}, 0) - X_{\alpha\beta} X_{\gamma\delta} D^{\alpha\beta\gamma\delta}(X_{\alpha\beta}, 0) \right. \\ & \left. + 2X_{\alpha\beta} \frac{\partial V^{\alpha\beta}}{\partial \varphi_{22}}(X_{\alpha\beta}, 0) + 2X_{\alpha\beta} \frac{\partial Y^{\alpha\beta}}{\partial \varphi_{11}}(X_{\alpha\beta}, 0) \right]. \end{aligned} \quad (4.16i)$$

In order to derive the components above, we have made use of the identity

$$\theta_{I\alpha} \theta_{I\beta} = -iX^+(\theta_I)^2 X_{\alpha\beta} \quad (4.17)$$

which is valid on the light-cone for each I . Note that a similar identity does not hold for $\theta_{I\alpha} \theta_{J\beta}$ if $I \neq J$. For this reason, and in order to simplify the formulas above, we have introduced an auxiliary, or average, variable $Z_{\text{aver}}^{\alpha\beta}$, defined as:

$$\theta_{1\alpha} \theta_{2\beta} Z^{\alpha\beta}(X, \varphi) \equiv -iX^+(\theta_1 \cdot \theta_2) X_{\alpha\beta} Z_{\text{aver}}^{\alpha\beta}(X, \varphi). \quad (4.18)$$

The quantity $Z_{\text{aver}}^{\alpha\beta}$ makes the scaling properties of the component explicit. It appears in (4.16d) and (4.16i) and should be viewed as a shortcut of more lengthy expressions corresponding to the expansion:

$$\begin{aligned} \theta_{1\alpha} \theta_{2\beta} Z^{\alpha\beta}(X, \varphi) &= (X^+)^2 \theta_{1a} \theta_{2b} \left[Z^{ab}(X, \varphi) + iy^\mu (\bar{\sigma}_\mu)^{\dot{b}b} Z^a_{\dot{b}}(X, \varphi) \right. \\ & \quad \left. + iy^\mu (\bar{\sigma}_\mu)^{\dot{a}a} Z_{\dot{a}}^b(X, \varphi) - y^\mu y^\nu (\bar{\sigma}_\mu)^{\dot{a}a} (\bar{\sigma}_\nu)^{\dot{b}b} Z_{\dot{a}\dot{b}}(X, \varphi) \right] = \\ &= (X^+)^2 \theta_{1a} \theta_{2b} \left[Z^{ab}(X, \varphi) + iy^\mu (\bar{\sigma}_\mu)^{\dot{b}b} Z^a_{\dot{b}}(X, \varphi) + iy^\mu (\bar{\sigma}_\mu)^{\dot{a}a} Z_{\dot{a}}^b(X, \varphi) \right. \\ & \quad \left. + \frac{1}{2} \left(y^2 \epsilon^{\dot{a}\dot{b}} \epsilon^{ab} - 4y_\mu y_\nu (\epsilon \sigma^{\lambda\mu})^{ab} (\bar{\sigma}^{\lambda\nu} \epsilon)^{\dot{a}\dot{b}} \right) Z_{\dot{a}\dot{b}}(X, \varphi) \right]. \end{aligned} \quad (4.19)$$

In going from the first equality to the second we have exploited the fact that the product $y^\mu y^\nu$ is symmetric in $\mu\nu$ and hence we can use the symmetric version of the product of two (barred) sigma matrices (see Appendix B). The r.h.s. of (4.18) is in components:

$$-iX^+\theta_1 \cdot \theta_2 [X_{ab}Z_{\text{aver}}^{ab}(X, \varphi) + X_a{}^{\dot{b}}Z_{\text{aver}}{}^a{}_{\dot{b}}(X, \varphi) + X^{\dot{a}}{}_{\dot{b}}Z_{\text{aver}}{}^{\dot{b}}{}_a(X, \varphi) + X^{\dot{a}\dot{b}}Z_{\text{aver}}{}_{\dot{a}\dot{b}}(X, \varphi)] \quad (4.20)$$

and hence one can derive the components of Z_{aver} , using (3.19) for the field $X_{\alpha\beta}$. Expanding now both sides in φ_{IJ} , we are left with ordinary fields depending on $X_{\alpha\beta}$ only. We obtain:

$$\begin{aligned} \theta_{1\alpha}\theta_{2\beta}Z^{\alpha\beta}(X, 0) &= (X^+)^2\theta_{1a}\theta_{2b}\left[Z^{ab}(X, 0) \right. \\ &\quad + iy^\mu(\bar{\sigma}_\mu)^{\dot{b}b}Z^a{}_{\dot{b}}(X, 0) + iy^\mu(\bar{\sigma}_\mu)^{\dot{a}a}Z_{\dot{a}}{}^b(X, 0) \\ &\quad \left. + \frac{1}{2}\left(y^2\epsilon^{\dot{a}\dot{b}}\epsilon^{ab} - 4y_\mu y_\nu(\epsilon\sigma^{\lambda\mu})^{ab}(\bar{\sigma}^{\lambda\nu}\epsilon)^{\dot{a}\dot{b}}\right)Z_{\dot{a}\dot{b}}(X, 0)\right] \end{aligned} \quad (4.21)$$

$$\begin{aligned} \theta_{1\alpha}\theta_{2\beta}4iX^+\theta_1 \cdot \theta_2 \frac{\partial Z^{\alpha\beta}}{\partial \varphi_{12}}(X, 0) &= (X^+)^2\theta_{1a}\theta_{2b}\left[4iX^+\theta_1 \cdot \theta_2 \left(\frac{\partial Z^{ab}}{\partial \varphi_{12}}(X, 0) \right. \right. \\ &\quad + iy^\mu(\bar{\sigma}_\mu)^{\dot{b}b}\frac{\partial Z^a{}_{\dot{b}}}{\partial \varphi_{12}}(X, 0) + iy^\mu(\bar{\sigma}_\mu)^{\dot{a}a}\frac{\partial Z_{\dot{a}}{}^b}{\partial \varphi_{12}}(X, 0) \\ &\quad \left. \left. + \frac{1}{2}\left(y^2\epsilon^{\dot{a}\dot{b}}\epsilon^{ab} - 4y_\mu y_\nu(\epsilon\sigma^{\lambda\mu})^{ab}(\bar{\sigma}^{\lambda\nu}\epsilon)^{\dot{a}\dot{b}}\right)\frac{\partial Z_{\dot{a}\dot{b}}}{\partial \varphi_{12}}(X, 0)\right)\right]. \end{aligned} \quad (4.22)$$

This modifies (4.16d) and (4.16i), which explicitly read:

$$\begin{aligned} Z^{ab}(y) &= (X^+)^{\Delta+1}\left[-4i\epsilon^{ab}\frac{\partial A}{\partial \varphi_{12}}(X, 0) \right. \\ &\quad + X^+\left[Z^{ab}(X, 0) + iy^\mu(\bar{\sigma}_\mu)^{\dot{b}b}Z^a{}_{\dot{b}}(X, 0) + iy^\mu(\bar{\sigma}_\mu)^{\dot{a}a}Z_{\dot{a}}{}^b(X, 0) \right. \\ &\quad \left. \left. + \frac{1}{2}\left(y^2\epsilon^{\dot{a}\dot{b}}\epsilon^{ab} - 4y_\mu y_\nu(\epsilon\sigma^{\lambda\mu})^{ab}(\bar{\sigma}^{\lambda\nu}\epsilon)^{\dot{a}\dot{b}}\right)Z_{\dot{a}\dot{b}}(X, 0)\right]\right] \end{aligned} \quad (4.23)$$

$$\begin{aligned} D(y) &= (X^+)^{\Delta+2}\left[2\frac{\partial^2 A}{\partial \varphi_{12}^2}(X, 0) - X_{\alpha\beta}X_{\gamma\delta}D^{\alpha\beta\gamma\delta}(X, 0) + 2X_{\alpha\beta}\frac{\partial V^{\alpha\beta}}{\partial \varphi_{22}}(X, 0) + 2X_{\alpha\beta}\frac{\partial Y^{\alpha\beta}}{\partial \varphi_{11}}(X, 0) \right. \\ &\quad - i\epsilon_{ab}X^+\left[\frac{\partial Z^{ab}}{\partial \varphi_{12}}(X, 0) + iy^\mu(\bar{\sigma}_\mu)^{\dot{b}b}\frac{\partial Z^a{}_{\dot{b}}}{\partial \varphi_{12}}(X, 0) + iy^\mu(\bar{\sigma}_\mu)^{\dot{a}a}\frac{\partial Z_{\dot{a}}{}^b}{\partial \varphi_{12}}(X, 0) \right. \\ &\quad \left. \left. + \frac{1}{2}\left(y^2\epsilon^{\dot{a}\dot{b}}\epsilon^{ab} - 4y_\mu y_\nu(\epsilon\sigma^{\lambda\mu})^{ab}(\bar{\sigma}^{\lambda\nu}\epsilon)^{\dot{a}\dot{b}}\right)\frac{\partial Z_{\dot{a}\dot{b}}}{\partial \varphi_{12}}(X, 0)\right]\right]. \end{aligned} \quad (4.24)$$

Using (4.9) we can determine the superconformal transformations for the components of the $\mathcal{N} = 2$ superfield. We obtain:

$$\delta A = \sqrt{2}(\epsilon_1 - iy_\mu \bar{\eta}^1 \bar{\sigma}^\mu) \cdot \lambda + \sqrt{2}(\epsilon_2 - iy_\mu \bar{\eta}^2 \bar{\sigma}^\mu) \cdot \chi \quad (4.25a)$$

$$\begin{aligned} \delta \lambda^a &= 2\sqrt{2}\Delta\eta^{1a}A + \sqrt{2}\epsilon^{ab}\left(i(\sigma^\mu)_{b\dot{a}}\bar{\epsilon}^{1\dot{a}} - y_\nu(\sigma^\mu)_{b\dot{a}}(\bar{\sigma}^\nu)^{\dot{a}c}\eta_c^1\right)\partial_\mu A \\ &\quad + \sqrt{2}\left(\epsilon_1^a - iy_\mu(\bar{\sigma}^\mu)^{\dot{a}a}\bar{\eta}_{\dot{a}}^1\right)B + \frac{\sqrt{2}}{2}\left(\epsilon_2^c - iy_\mu(\bar{\sigma}^\mu)^{\dot{a}c}\bar{\eta}_{\dot{a}}^2\right)\epsilon_{cb}Z^{ab} \end{aligned} \quad (4.25b)$$

$$\begin{aligned} \delta \chi^a &= 2\sqrt{2}\Delta\eta^{2a}A + \sqrt{2}\epsilon^{ab}\left(i(\sigma^\mu)_{b\dot{a}}\bar{\epsilon}^{2\dot{a}} - y_\nu(\sigma^\mu)_{b\dot{a}}(\bar{\sigma}^\nu)^{\dot{a}c}\eta_c^2\right)\partial_\mu A \\ &\quad + \sqrt{2}\left(\epsilon_2^a - iy_\mu(\bar{\sigma}^\mu)^{\dot{a}a}\bar{\eta}_{\dot{a}}^2\right)C - \frac{\sqrt{2}}{2}\left(\epsilon_1^c - iy_\mu(\bar{\sigma}^\mu)^{\dot{a}c}\bar{\eta}_{\dot{a}}^1\right)\epsilon_{cb}Z^{ba} \end{aligned} \quad (4.25c)$$

$$\begin{aligned} \delta Z^{ab} &= -4\sqrt{2}\Delta\eta^{1a}\chi^b - 4\sqrt{2}\Delta\lambda^a\eta^{2b} + \sqrt{2}\epsilon^{ac}\left(2i(\sigma^\mu)_{c\dot{a}}\bar{\epsilon}^{1\dot{a}} - 2y_\nu(\sigma^\mu)_{c\dot{a}}(\bar{\sigma}^\nu)^{\dot{a}d}\eta_d^1\right)\partial_\mu\chi^b \\ &\quad - \sqrt{2}\epsilon^{bc}\left(2i(\sigma^\mu)_{c\dot{a}}\bar{\epsilon}^{2\dot{a}} - 2y_\nu(\sigma^\mu)_{c\dot{a}}(\bar{\sigma}^\nu)^{\dot{a}d}\eta_d^2\right)\partial_\mu\lambda^a - 2\left(\epsilon_1^a - iy_\mu(\bar{\sigma}^\mu)^{\dot{a}a}\bar{\eta}_{\dot{a}}^1\right)E^b \\ &\quad + 2\left(\epsilon_2^a - iy_\mu(\bar{\sigma}^\mu)^{\dot{a}a}\bar{\eta}_{\dot{a}}^2\right)F^b - 4\sqrt{2}\epsilon^{ab}\eta^2 \cdot \lambda - 4\sqrt{2}\epsilon^{ab}\eta^1 \cdot \chi \end{aligned} \quad (4.25d)$$

$$\delta B = 2\sqrt{2}\Delta\eta^1 \cdot \lambda + \sqrt{2}\left(i(\sigma^\mu)_{b\dot{a}}\bar{\epsilon}^{1\dot{a}} - y_\nu(\sigma^\mu)_{b\dot{a}}(\bar{\sigma}^\nu)^{\dot{a}c}\eta_c^1\right)\partial_\mu\lambda^b + \left(\epsilon_2^a - iy_\mu(\bar{\sigma}^\mu)^{\dot{a}a}\bar{\eta}_{\dot{a}}^2\right)E_a \quad (4.25e)$$

$$\delta C = 2\sqrt{2}\Delta\eta^2 \cdot \chi + \sqrt{2}\left(i(\sigma^\mu)_{b\dot{a}}\bar{\epsilon}^{2\dot{a}} - y_\nu(\sigma^\mu)_{b\dot{a}}(\bar{\sigma}^\nu)^{\dot{a}c}\eta_c^2\right)\partial_\mu\chi^b + \left(\epsilon_1^a - iy_\mu(\bar{\sigma}^\mu)^{\dot{a}a}\bar{\eta}_{\dot{a}}^1\right)F_a \quad (4.25f)$$

$$\begin{aligned} \delta E^a &= (4 - 2\Delta)\eta_b^1 Z^{ba} - 2Z^{ab}\eta_b^1 + (4\Delta + 4)\eta^{2a}B - \left(i(\sigma^\mu)_{b\dot{a}}\bar{\epsilon}^{1\dot{a}} - y_\nu(\sigma^\mu)_{b\dot{a}}(\bar{\sigma}^\nu)^{\dot{a}c}\eta_c^1\right)\partial_\mu Z^{ba} \\ &\quad + \epsilon^{ab}\left(2i(\sigma^\mu)_{b\dot{a}}\bar{\epsilon}^{2\dot{a}} - 2y_\nu(\sigma^\mu)_{b\dot{a}}(\bar{\sigma}^\nu)^{\dot{a}c}\eta_c^2\right)\partial_\mu B + 2\left(\epsilon_2^a - iy_\mu(\bar{\sigma}^\mu)^{\dot{a}a}\bar{\eta}_{\dot{a}}^2\right)D \end{aligned} \quad (4.25g)$$

$$\begin{aligned} \delta F^a &= 2\eta_b^2 Z^{ba} - (4 - 2\Delta)Z^{ab}\eta_b^2 + (4\Delta + 4)\eta^{1a}C + \left(i(\sigma^\mu)_{b\dot{a}}\bar{\epsilon}^{2\dot{a}} - y_\nu(\sigma^\mu)_{b\dot{a}}(\bar{\sigma}^\nu)^{\dot{a}c}\eta_c^2\right)\partial_\mu Z^{ba} \\ &\quad + \epsilon^{ab}\left(2i(\sigma^\mu)_{b\dot{a}}\bar{\epsilon}^{1\dot{a}} - 2y_\nu(\sigma^\mu)_{b\dot{a}}(\bar{\sigma}^\nu)^{\dot{a}c}\eta_c^1\right)\partial_\mu C + 2\left(\epsilon_1^a - iy_\mu(\bar{\sigma}^\mu)^{\dot{a}a}\bar{\eta}_{\dot{a}}^1\right)D \end{aligned} \quad (4.25h)$$

$$\begin{aligned} \delta D &= (2 - 2\Delta)\eta^1 \cdot F + (2 - 2\Delta)\eta^2 \cdot E - \left(i(\sigma^\mu)_{b\dot{a}}\bar{\epsilon}^{1\dot{a}} - y_\nu(\sigma^\mu)_{b\dot{a}}(\bar{\sigma}^\nu)^{\dot{a}c}\eta_c^1\right)\partial_\mu F^b \\ &\quad - \left(i(\sigma^\mu)_{b\dot{a}}\bar{\epsilon}^{2\dot{a}} - y_\nu(\sigma^\mu)_{b\dot{a}}(\bar{\sigma}^\nu)^{\dot{a}c}\eta_c^2\right)\partial_\mu E^b. \end{aligned} \quad (4.25i)$$

Special conformal transformations correspond to $\epsilon_I^a = 0$, as it was written in [21]. By setting $\epsilon_I^a = 0$, we see that at the origin the $\theta_I = \bar{\theta}_I = 0$ component is invariant, namely $\delta A(0) = 0$, and hence $A(0)$ is annihilated by the special superconformal generators $S_a^I, \bar{S}_{\dot{a}}^I$, defined by

$$Q_\alpha^I = \frac{i}{2} \begin{pmatrix} -Q_a^I \\ \bar{S}_{\dot{a}}^I \end{pmatrix} \quad \text{and} \quad \bar{Q}_I^\alpha = \frac{i}{2} \begin{pmatrix} -S_I^a \\ \bar{Q}_{\dot{a}}^I \end{pmatrix}. \quad (4.26)$$

This can be checked by using the appropriate generalizations of those operators from the $\mathcal{N} = 1$ case of [21], which tells us that $\Phi(y, \theta_I)$ is a chiral field generated by a chiral primary operator A . This is a generic property of the chiral sector. In fact, using the transformation rules (3.24), for special conformal transformations, $\delta A \equiv \delta\Phi| = -\delta\theta_I^a \frac{\partial\Phi}{\partial\theta_I^a}|$ (the vertical bar denoting the $\theta_I = 0$ component of the r.h.s., which corresponds to taking the spinorial θ -component of Φ) will always be proportional to y^μ and hence will vanish at the origin.

5 Conclusion

In this paper we generalized the construction of superembedding methods for 4d $\mathcal{N} = 1$ in [21] to embeddings of \mathcal{N} -extended superconformal field theories. In this way, the superconformal group acts linearly on the coordinates of the ambient space, the conformal symmetry is manifest and moreover the method is valid for any conformally flat space, not just Minkowski. We have considered explicitly the case of $\mathcal{N} = 2$ chiral superfields in four dimensions and concluded that any conformal chiral superfield of the \mathcal{N} -extended supersymmetry is generated by a chiral primary operator sitting in its highest component. We have also noted a correspondence between the number of component fields of a certain type in any chiral multiplet of the \mathcal{N} -extended supersymmetry and the entries of the Pascal pyramid at layer \mathcal{N} . This correspondence has been explicitly showed in the cases of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetry.

We have not considered correlation functions, that for $\mathcal{N} = 1$ were presented in [21], neither non-holomorphic operators and higher-rank tensors, which among other things are relevant in AdS/CFT and its applications. These points are left as open questions.

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Appendix

A Explicit $SU(2,2)$ transformations of $X_{\alpha\beta}$

For the six-dimensional vector $X^m = (X^+, X^\mu, X^-)$ the transformations under the conformal group $SO(4,2)$ are $\delta X^m = \omega^{mn} X_n$, with ω^{mn} anti-symmetric. The four-dimensional coordinates x^μ are related to the six-dimensional ones by solving the light-cone constraint:

$$X^+ = \text{constant}, \quad x^\mu = \frac{X^\mu}{X^+}, \quad X^- = -X^+ x^2. \quad (\text{A.1})$$

Using the conventions of [21], since⁸ $X_{\alpha\beta} = \frac{1}{2} X_m \tilde{\Gamma}_{\alpha\beta}^m$ in $SU(2,2)$ notation, these transformations are:

$$\begin{aligned} \delta X_{\alpha\beta} &= \frac{1}{2} \delta X_m \tilde{\Gamma}_{\alpha\beta}^m = \frac{1}{2} \delta X_+ \tilde{\Gamma}_{\alpha\beta}^+ + \frac{1}{2} \delta X_\mu \tilde{\Gamma}_{\alpha\beta}^\mu + \frac{1}{2} \delta X_- \tilde{\Gamma}_{\alpha\beta}^- = \\ &= \frac{1}{4} \delta X^- \tilde{\Gamma}_{\alpha\beta}^+ + \frac{1}{2} \delta X^\nu \eta_{\mu\nu} \tilde{\Gamma}_{\alpha\beta}^\mu + \frac{1}{4} \delta X^+ \tilde{\Gamma}_{\alpha\beta}^- = \\ &= \begin{pmatrix} i\frac{1}{2} \delta X^+ \epsilon_{ab} & \frac{1}{2} \eta_{\mu\nu} \delta X^\nu (\sigma^\mu)_{ad} \epsilon^{db} \\ -\frac{1}{2} \eta_{\mu\nu} \delta X^\nu (\bar{\sigma}^\mu)^{\dot{a}d} \epsilon_{db} & i\frac{1}{2} \delta X^- \epsilon^{\dot{a}\dot{b}} \end{pmatrix}. \end{aligned} \quad (\text{A.3})$$

⁸The index structure of the tensors $X_{\alpha\beta}$ and $\bar{X}^{\alpha\beta}$ is

$$X_{\alpha\beta} = \begin{pmatrix} X_{ab} & X_a{}^{\dot{b}} \\ X^{\dot{a}}{}_b & X^{\dot{a}\dot{b}} \end{pmatrix} \quad \text{and} \quad \bar{X}^{\alpha\beta} = \begin{pmatrix} \bar{X}^{ab} & \bar{X}^a{}_{\dot{b}} \\ \bar{X}_{\dot{a}}{}^b & \bar{X}_{\dot{a}\dot{b}} \end{pmatrix}. \quad (\text{A.2})$$

In the next subsections, we consider the single conformal transformations explicitly. We will also apply these transformations to the origin (3.7) and will find the summarizing table 1, that is given here for convenience.

Table 1: Summary of $SU(2, 2)$ transformations of the origin

Translations	$\delta \hat{X}_{\alpha\beta} = \frac{1}{4} \eta_{\mu\nu} \omega^{\nu-} X^+ \cdot \begin{pmatrix} 0 & (\sigma^\mu)_{ad} \epsilon^{db} \\ -(\bar{\sigma}^\mu)^{\dot{a}d} \epsilon_{db} & 0 \end{pmatrix}$
Lorentz	$\delta \hat{X}_{\alpha\beta} = 0$
Special Conformal	$\delta \hat{X}_{\alpha\beta} = 0$
Dilatations	$\delta \hat{X}_{\alpha\beta} = \frac{1}{4} \omega^{+-} \cdot \begin{pmatrix} i\epsilon_{ab} X^+ & 0 \\ 0 & 0 \end{pmatrix}$

A.1 Translations

The only non-zero parameter is $\omega^{\mu-}$. Hence

$$\delta X^+ = \omega^{+n} X_n = 0 \quad (\text{A.4a})$$

$$\delta X^\mu = \omega^{\mu n} X_n = \omega^{\mu-} X_- = \frac{1}{2} \omega^{\mu-} X^+ \quad (\text{A.4b})$$

$$\delta X^- = \omega^{-n} X_n = -\omega^{\mu-} X_\mu = -\eta_{\mu\nu} \omega^{\mu-} X^\nu. \quad (\text{A.4c})$$

The second line gives the four-dimensional translation

$$\delta x^\mu = \frac{1}{2} \omega^{\mu-} \quad (\text{A.5})$$

and consistently the third line gives $\delta x^2 = \eta_{\mu\nu} \omega^{\mu-} x^\nu$. Using (A.3) one has:

$$\delta X_{\alpha\beta} = \frac{1}{2} \eta_{\mu\nu} \omega^{\nu-} \cdot \begin{pmatrix} 0 & \frac{1}{2} X^+ (\sigma^\mu)_{ad} \epsilon^{db} \\ -\frac{1}{2} X^+ (\bar{\sigma}^\mu)^{\dot{a}d} \epsilon_{db} & -i X^\mu \end{pmatrix}. \quad (\text{A.6})$$

When evaluated at the origin, this becomes

$$\delta \hat{X}_{\alpha\beta} = \frac{1}{4} \eta_{\mu\nu} \omega^{\nu-} X^+ \cdot \begin{pmatrix} 0 & (\sigma^\mu)_{ad} \epsilon^{db} \\ -(\bar{\sigma}^\mu)^{\dot{a}d} \epsilon_{db} & 0 \end{pmatrix}. \quad (\text{A.7})$$

A.2 Lorentz transformations

The only non-zero parameter is $\omega^{\mu\nu}$. This implies:

$$\delta X^+ = \omega^{+n} X_n = 0 \quad (\text{A.8a})$$

$$\delta X^\mu = \omega^{\mu\nu} X_\nu = \omega^{\mu\nu} \eta_{\nu\rho} X^\rho \quad (\text{A.8b})$$

$$\delta X^- = \omega^{-n} X_n = 0. \quad (\text{A.8c})$$

The second line implies the four-dimensional Lorentz transformation

$$\delta x^\mu = \omega^{\mu\nu} x_\nu \quad (\text{A.9})$$

and consistently the third line gives $\delta x^2 = 0$. In $SU(2, 2)$ notation, one has

$$\delta X_{\alpha\beta} = \frac{1}{2} \eta_{\mu\nu} \omega^{\nu\rho} X_\rho \cdot \begin{pmatrix} 0 & (\sigma^\mu)_{ad} \epsilon^{db} \\ -(\bar{\sigma}^\mu)^{\dot{a}d} \epsilon_{db} & 0 \end{pmatrix}, \quad (\text{A.10})$$

which vanishes at the origin,

$$\delta \hat{X}_{\alpha\beta} = 0. \quad (\text{A.11})$$

A.3 Special conformal transformations

Here $\omega^{\mu+}$ is non-zero and

$$\delta X^+ = \omega^{+n} X_n = \omega^{+\mu} X_\mu = -\omega^{\mu+} X_\mu \quad (\text{A.12a})$$

$$\delta X^\mu = \omega^{\mu n} X_n = \omega^{\mu+} X_+ = \frac{1}{2} \omega^{\mu+} X^- \quad (\text{A.12b})$$

$$\delta X^- = \omega^{-n} X_n = 0. \quad (\text{A.12c})$$

The first two lines combined give the four-dimensional special conformal transformation

$$\delta x^\mu = x^2 b^\mu - 2x^\mu (b \cdot x), \quad (\text{A.13})$$

with $b^\mu = -\frac{1}{2} \omega^{\mu+}$. In $SU(2, 2)$ notation, one has

$$\delta X_{\alpha\beta} = \frac{1}{2} \eta_{\mu\nu} \omega^{\nu+} \cdot \begin{pmatrix} -iX_\mu & \frac{1}{2} X^- (\sigma^\mu)_{ad} \epsilon^{db} \\ -\frac{1}{2} X^- (\bar{\sigma}^\mu)^{\dot{a}d} \epsilon_{db} & 0 \end{pmatrix}, \quad (\text{A.14})$$

which vanishes at the origin,

$$\delta \hat{X}_{\alpha\beta} = 0. \quad (\text{A.15})$$

A.4 Dilatations

The non-vanishing parameter is ω^{+-} and

$$\delta X^+ = \omega^{+n} X_n = \omega^{+-} X_- = \frac{1}{2} \omega^{+-} X^+ \quad (\text{A.16a})$$

$$\delta X^\mu = \omega^{\mu n} X_n = 0 \quad (\text{A.16b})$$

$$\delta X^- = \omega^{-n} X_n = \omega^{-+} X_+ = -\frac{1}{2} \omega^{+-} X^-. \quad (\text{A.16c})$$

The first two lines combined give the four-dimensional scale transformation

$$\delta x^\mu = -\frac{1}{2} \omega^{+-} x^\mu, \quad (\text{A.17})$$

while the third line gives consistently $\delta x^2 = -\omega^{+-} x^2$. In spinor notation,

$$\delta X_{\alpha\beta} = \frac{1}{4} \omega^{+-} \cdot \begin{pmatrix} iX^+ \epsilon_{ab} & 0 \\ 0 & iX^- \epsilon^{\dot{a}\dot{b}} \end{pmatrix}. \quad (\text{A.18})$$

At the origin this becomes

$$\delta \hat{X}_{\alpha\beta} = \frac{1}{4} \omega^{+-} \cdot \begin{pmatrix} iX^+ \epsilon_{ab} & 0 \\ 0 & 0 \end{pmatrix}. \quad (\text{A.19})$$

B Notation and identities

In this appendix we write down some spinor identities that are used in the calculations of the main text. Our conventions for the spinors are the same as in [21], in particular:

$$V_\alpha = \begin{pmatrix} \psi_a \\ \bar{\chi}^{\dot{a}} \end{pmatrix} \iff \bar{V}^\alpha = \bar{V}_{\dot{\alpha}} A^{\dot{\alpha}\alpha} = (\chi^a, \bar{\psi}_{\dot{a}}), \quad (\text{B.1})$$

where

$$A^{\dot{\alpha}\beta} = \begin{pmatrix} 0 & \delta^{\dot{a}b} \\ \delta_a^{\dot{b}} & 0 \end{pmatrix}. \quad (\text{B.2})$$

The gamma matrices are given by:

$$\Gamma^{+\alpha\beta} = \begin{pmatrix} 2i\epsilon^{ab} & 0 \\ 0 & 0 \end{pmatrix}, \quad \Gamma^{-\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 2i\epsilon_{\dot{a}\dot{b}} \end{pmatrix}, \quad \Gamma^{\mu\alpha\beta} = \begin{pmatrix} 0 & \bar{\sigma}^{\mu\dot{a}a}\epsilon_{\dot{b}b} \\ -\sigma^\mu_{\dot{a}a}\epsilon^{db} & 0 \end{pmatrix} \quad (\text{B.3})$$

and

$$\tilde{\Gamma}_{\alpha\beta}^+ = \begin{pmatrix} 0 & 0 \\ 0 & 2i\epsilon^{\dot{a}\dot{b}} \end{pmatrix}, \quad \tilde{\Gamma}_{\alpha\beta}^- = \begin{pmatrix} 2i\epsilon_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\Gamma}_{\alpha\beta}^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\dot{a}a}\epsilon^{\dot{b}b} \\ -\bar{\sigma}^{\mu\dot{a}d}\epsilon_{db} & 0 \end{pmatrix} \quad (\text{B.4})$$

The $SO(4, 2)$ coordinate X^m and the anti-symmetric $SU(2, 2)$ tensor $X_{\alpha\beta}$ are related by

$$X^m = \frac{1}{2}X_{\alpha\beta}\Gamma^{m\alpha\beta} = \frac{1}{2}X^{\alpha\beta}\tilde{\Gamma}_{\alpha\beta}^m \quad (\text{B.5})$$

and

$$X_{\alpha\beta} = \frac{1}{2}X_m\tilde{\Gamma}_{\alpha\beta}^m, \quad X^{\alpha\beta} = \frac{1}{2}X_m\Gamma^{m\alpha\beta}. \quad (\text{B.6})$$

As it is standard in supersymmetry, for each I , the product $\bar{\theta}^{I\dot{a}}\theta_I^b$ must be proportional to $(\bar{\sigma}^\mu)^{\dot{a}b}$, i.e. $2\bar{\theta}^{I\dot{a}}\theta_I^b = A^\mu(\bar{\sigma}_\mu)^{\dot{a}b}$, where the proportionality constant A^μ is fixed by multiplying both sides by $(\sigma^\nu)_{b\dot{a}}$ and summing over b and \dot{a} (which gives $\text{Tr}(\bar{\sigma}^\mu\sigma^\nu) = -2\eta^{\mu\nu}$). After anti-commuting the theta's, one finds $A^\mu = \theta_I\sigma^\mu\bar{\theta}^I$ and hence, for each I :

$$\bar{\theta}^{I\dot{a}}\theta_I^b = \frac{1}{2}\theta_I\sigma^\mu\bar{\theta}^I(\bar{\sigma}_\mu)^{\dot{a}b}. \quad (\text{B.7})$$

This expression was used in (3.15). One can make similar manipulations in extended supersymmetry and write (repeated indices are not summed):

$$\theta_{Ia}\theta_{Ib} = \frac{1}{2}(\theta_I)^2\epsilon_{ab} \quad (\text{B.8})$$

$$\theta_I \cdot \theta_J = -\epsilon^{ab}\theta_{Ia}\theta_{Jb} \quad (\text{B.9})$$

$$\theta_{Ia}(\theta_I \cdot \theta_J) = \frac{1}{2}(\theta_I)^2\theta_{Ja} \quad (\text{B.10})$$

$$(\theta_I \cdot \theta_J)^2 = -\frac{1}{2}(\theta_I)^2(\theta_J)^2 \quad (\text{B.11})$$

$$\theta_{Ia}\theta_{Jb}(\theta_I \cdot \theta_J) = -\frac{1}{4}(\theta_I)^2(\theta_J)^2\epsilon_{ab}, \quad (\text{B.12})$$

which were used, for example, in (4.16) with $I, J = 1, 2$.

Our conventions for the sigma matrices are as in [21] and [23]. In particular, $\sigma^\mu = \{-1, \sigma^i\}$ and $\bar{\sigma}^\mu = \{-1, -\sigma^i\}$. These are related by

$$(\bar{\sigma}^\mu)^{\dot{a}a} = \epsilon^{\dot{a}b} \epsilon^{ab} (\sigma^\mu)_{bb}. \quad (\text{B.13})$$

This implies that

$$(\sigma^\mu)_{ad} \epsilon^{\dot{d}b} = (\bar{\sigma}^\mu)^{\dot{b}d} \epsilon_{da} \quad \text{and} \quad (\sigma^\mu)_{a\dot{a}} = \epsilon_{ad} \epsilon_{\dot{a}d} (\bar{\sigma}^\mu)^{\dot{d}d}. \quad (\text{B.14})$$

The former equation was used in (3.19). There, we also used the following relation:

$$y^\mu y^\nu (\bar{\sigma}_\mu)^{\dot{a}c} (\bar{\sigma}_\nu)^{\dot{b}d} \epsilon_{cd} = y^\mu y^\nu \eta_{\mu\nu} \epsilon^{\dot{a}b}. \quad (\text{B.15})$$

This comes from the more generic relation regarding the product of two sigma matrices, which can be expressed in terms of the generators $(\sigma^{mn})_a^b$ and $(\bar{\sigma}^{mn})^{\dot{a}}_{\dot{b}}$ of the Lorentz group in the spinor representation, which are anti-symmetric in the indices m and n (see below). However, the symmetric part in the space-time indices and the anti-symmetric part in the spinorial indices is fully specified by the metric tensor and the epsilon tensor as:

$$\bar{\sigma}_{(\mu}^{\dot{a}[c} \bar{\sigma}_{\nu)}^{\dot{b}]d} = -\frac{1}{2} \eta_{\mu\nu} \epsilon^{\dot{a}b} \epsilon^{cd}. \quad (\text{B.16})$$

A similar formula holds for the anti-symmetric combination of the dotted indices as well as for the product of two (unbarred) sigma matrices.

Using the relations above, it is straightforward to check that:

$$\bar{\epsilon} \sigma^\mu \theta = -\theta \sigma^\mu \bar{\epsilon} \quad \text{and} \quad \eta \sigma^\nu \bar{\sigma}^\mu \theta = \theta \sigma^\mu \bar{\sigma}^\nu \eta. \quad (\text{B.17})$$

These identities were used in (3.24). If we only look at the symmetric part in the space-time indices, instead, we have to consider the more general expression

$$\bar{\sigma}_{(\mu}^{\dot{a}a} \bar{\sigma}_{\nu)}^{\dot{b}b} = \frac{1}{2} [-\eta_{\mu\nu} \epsilon^{\dot{a}b} \epsilon^{ab} + 4(\epsilon \sigma^{\lambda\mu})^{ab} (\bar{\sigma}^{\lambda\nu} \epsilon)^{\dot{a}b}], \quad (\text{B.18})$$

as it appears in [23], which was used in (4.16).

C Barred 4D superspace

In the translations given in (2.28) we need to know the inverse of (3.17). Because of (3.13), the inverse is a product of the two commuting transformations

$$\mathcal{U}^{-1}(x^\mu, \{\theta_{Ia}\}, \{\bar{\theta}^{J\dot{a}}\})_A^B = \mathcal{U}^{-1}(x^\mu, 0, 0)_A^C \mathcal{U}^{-1}(0, \{\theta_{Ia}\}, \{\bar{\theta}^{J\dot{a}}\})_C^B. \quad (\text{C.1})$$

The space-time part is simply obtained by replacing $x^\mu \rightarrow -x^\mu$ in (3.14):

$$\mathcal{U}^{-1}(x^\mu, 0, 0)_A^B = \begin{pmatrix} \left(e^{+ix^\mu \Sigma^{\mu+}}\right)_\alpha^\beta & 0 \\ 0 & \delta_I^J \end{pmatrix} = \begin{pmatrix} \delta_a^b & 0 & 0 \\ -ix^\mu (\bar{\sigma}_\mu)^{\dot{a}b} & \delta_{\dot{b}}^{\dot{a}} & 0 \\ 0 & 0 & \delta_I^J \end{pmatrix}. \quad (\text{C.2})$$

To compute the theta part, it is convenient to use the following expression for the inversion formula of block matrices:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{pmatrix}, \quad (\text{C.3})$$

with

- $(A - BD^{-1}C) = \begin{pmatrix} \delta_a^b & 0 \\ 2\sum_I \bar{\theta}^{I\dot{a}} \theta_I^b & \delta_{\dot{b}}^{\dot{a}} \end{pmatrix}$
- $(A - BD^{-1}C)^{-1} = \begin{pmatrix} \delta_a^b & 0 \\ -2\sum_I \bar{\theta}^{I\dot{a}} \theta_I^b & \delta_{\dot{b}}^{\dot{a}} \end{pmatrix}$
- $-(A - BD^{-1}C)^{-1}BD^{-1} = -\begin{pmatrix} 0 \\ 2i\bar{\theta}^{J\dot{a}} \end{pmatrix}$
- $-D^{-1}C(A - BD^{-1}C)^{-1} = -(2i\theta_I^b, 0)$
- $D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} = 1_{\mathcal{N} \times \mathcal{N}}$

Hence:

$$\mathcal{U}^{-1}(0, \{\theta_{Ia}\}, \{\bar{\theta}^{J\dot{a}}\})_A^B = \begin{pmatrix} \delta_a^b & 0 & 0 \\ -2\sum_I \bar{\theta}^{I\dot{a}} \theta_I^b & \delta_{\dot{b}}^{\dot{a}} & -2i\bar{\theta}^{J\dot{a}} \\ -2i\theta_I^b & 0 & \delta_I^J \end{pmatrix}. \quad (\text{C.4})$$

The complete inverse transformation is then the product of (C.2) and (C.4):

$$\mathcal{U}^{-1}(x^\mu, \theta_{Ia}, \bar{\theta}^{J\dot{a}})_A^B = \begin{pmatrix} \delta_a^b & 0 & 0 \\ -i\bar{y}^\mu (\bar{\sigma}_\mu)^{\dot{a}b} & \delta_{\dot{b}}^{\dot{a}} & -2i\bar{\theta}^{J\dot{a}} \\ -2i\theta_I^b & 0 & \delta_I^J \end{pmatrix}, \quad (\text{C.5})$$

where $\bar{y}^\mu = x^\mu - i\sum_I \theta_I \sigma^\mu \bar{\theta}^I = (y^\mu)^\dagger$ and we have again replaced, for each I , $\bar{\theta}^{I\dot{a}} \theta_I^b = \frac{1}{2} \theta_I \sigma^\mu \bar{\theta}^I (\bar{\sigma}_\mu)^{\dot{a}b}$. This is enough to derive the set of equations (3.21).

D Pascal Pyramid and $\mathcal{N} = 4$ chiral superfield

D.1 Pascal Pyramid

In number theory, in analogy to the binomial expansion, whose coefficients can be organized into a triangle, one can consider the trinomial:

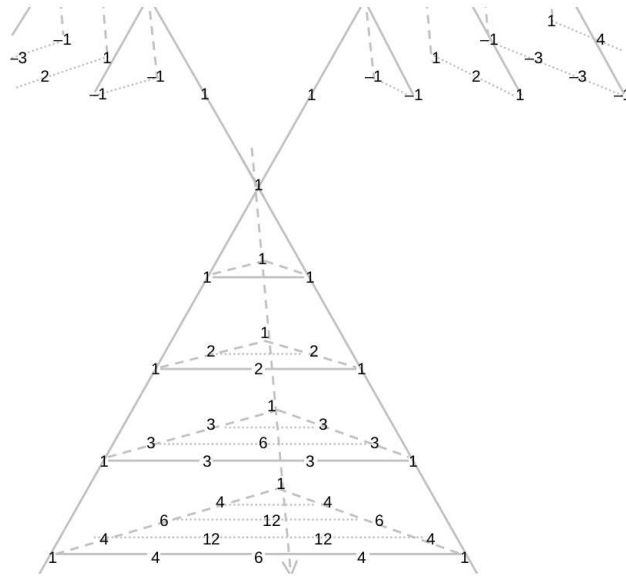
$$(a + b + c)^n = \sum_{m=0}^n \sum_{k=0}^m \binom{n}{m} \binom{m}{k} a^{n-m} b^{m-k} c^k \equiv \sum_{\substack{i,j,k \\ i+j+k=n}} \binom{n}{i,j,k} a^i b^j c^k, \quad (\text{D.1})$$

where the integer numbers appearing in the Pascal pyramid (tetrahedron) are just the coefficients $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$ of the expansion (see figure 1) of the trinomial. The power n indicates the layer of the pyramid.

The pyramid has a huge number of symmetries and interesting properties. For instance, by looking at specific examples or by using some combinatorics, one can check that there are $(n+1)(n+2)/2$ elements at each layer. Also, the sum of the entries of each layer is 3^n , as can be seen by taking $a = b = c = 1$ in (D.1). Moreover, by superimposing two consecutive layers, it is possible to prove that each entry at layer n is given by the sum of all the entries at layer $n-1$ that surround it. This is simply a property of the trinomial coefficients.

In number theory, one generalizes binomials and trinomials to multinomials and, in a similar way, we speak of an m -simplex, which generalizes triangles and tetrahedrons. A multinomial is by definition

$$(a_1 + \dots + a_m)^n = \sum_{\substack{k_1, \dots, k_m \\ k_1 + \dots + k_m = n}} \binom{n}{k_1, \dots, k_m} a_1^{k_1} \dots a_m^{k_m}, \quad (\text{D.2})$$



variables as

$$\begin{aligned}
\Phi(X_{\alpha\beta}, \theta_{I\alpha}, \varphi_{IJ}) = & A + \sum_I \theta_{I\alpha} \lambda^{I,\alpha} + \sum_{I<J} \theta_{I\alpha} \theta_{J\beta} E^{IJ,\alpha\beta} + \sum_I \theta_{I\alpha} \theta_{I\beta} E^{I,\alpha\beta} \\
& + \frac{1}{6} \sum_{I,J,K,L} \epsilon_{IJKL} \theta_{J\alpha} \theta_{K\beta} \theta_{L\gamma} \chi^{I,\alpha\beta\gamma} + \sum_{I \neq J} \theta_{J\alpha} \theta_{J\beta} \theta_{I\gamma} \chi^{IJ,\alpha\beta\gamma} \\
& + \theta_{1\alpha} \theta_{2\beta} \theta_{3\gamma} \theta_{4\delta} B^{\alpha\beta\gamma\delta} + \frac{1}{2} \sum_{I,J,K,L} \epsilon_{IJKL} \theta_{I\alpha} \theta_{J\beta} \theta_{K\gamma} \theta_{L\delta} \tilde{B}^{IL,\alpha\beta\gamma\delta} \\
& + \sum_{I<J} \theta_{I\alpha} \theta_{I\beta} \theta_{J\gamma} \theta_{J\delta} B^{IJ,\alpha\beta\gamma\delta} + \frac{1}{6} \sum_{I,J,K,L} \epsilon_{IJKL} \theta_{I\alpha} \theta_{I\beta} \theta_{J\gamma} \theta_{K\delta} \theta_{L\eta} \xi^{I,\alpha\beta\gamma\delta\eta} \\
& + \frac{1}{2} \sum_{I \neq J \neq K} \theta_{J\alpha} \theta_{J\beta} \theta_{K\gamma} \theta_{K\delta} \theta_{I\eta} \xi^{IJK,\alpha\beta\gamma\delta\eta} \\
& + \frac{1}{2} \sum_{\substack{I<J \\ I \neq J \neq K \neq L}} \theta_{I\alpha} \theta_{J\beta} \theta_{K\gamma} \theta_{K\delta} \theta_{L\eta} \theta_{L\epsilon} C^{IJKL,\alpha\beta\gamma\delta\eta\epsilon} \\
& + \frac{1}{6} \sum_{I \neq J \neq K} \theta_{I\alpha} \theta_{I\beta} \theta_{J\gamma} \theta_{J\delta} \theta_{K\eta} \theta_{K\epsilon} C^{IJK,\alpha\beta\gamma\delta\eta\epsilon} \\
& + \frac{1}{6} \sum_{I \neq J \neq K \neq L} \theta_{J\alpha} \theta_{J\beta} \theta_{K\gamma} \theta_{K\delta} \theta_{L\eta} \theta_{L\epsilon} \theta_{I\sigma} \omega^{I,\alpha\beta\gamma\delta\eta\epsilon\sigma} \\
& + \theta_{1\alpha} \theta_{1\beta} \theta_{2\gamma} \theta_{2\delta} \theta_{3\eta} \theta_{3\epsilon} \theta_{4\sigma} \theta_{4\rho} D^{\alpha\beta\gamma\delta\eta\epsilon\sigma\rho} + \dots
\end{aligned} \tag{D.4}$$

Here, I, J, \dots , etc. $= 1, \dots, 4$ can be raised/lowered with the flat (identity) metric, all the component fields are functions of $(X_{\alpha\beta}, \varphi_{IJ})$ and the dots in the end represent quantities that will vanish on the light-cone. The four-dimensional superfield as well as its components are defined on the light-cone and will be functions of $y^\mu = x^\mu + \sum_I \theta_I \sigma^\mu \bar{\theta}^I$ only. Moreover, the φ_{IJ} -dependence is removed by replacing $\varphi_{IJ} = 2iX^+ \theta_I \cdot \theta_J$ and expanding everything in θ_{Ia} . The result is:

$$\begin{aligned}
\Phi(y^\mu, \theta^{Ia}) \equiv & (X^+)^\Delta \Phi(X_{\alpha\beta}, \theta_{I\alpha}, \varphi_{IJ}) = \\
& A(y) + \sum_I \theta_{Ia} \Lambda^{I,a}(y) + \sum_{I<J} \theta_{Ia} \theta_{Jb} F^{IJ,ab}(y) + \sum_I \theta_I^2 F^I(y) \\
& + \frac{1}{6} \sum_{I,J,K,L} \epsilon_{IJKL} \theta_{Ja} \theta_{Kb} \theta_{Lc} \Upsilon^{I,abc}(y) + \sum_{I \neq J} \theta_{Ia} \theta_J^2 \Upsilon^{IJ,a}(y) + \theta_{1a} \theta_{2b} \theta_{3c} \theta_{4d} B^{abcd} \\
& + \frac{1}{2} \sum_{\substack{J<K \\ J \neq K \neq L}} \theta_{Ja} \theta_{Kb} \theta_L^2 B^{JKL,ab}(y) + \sum_{I<J} \theta_I^2 \theta_J^2 B^{(IJ)}(y) \\
& + \frac{1}{6} \sum_{I,J,K,L} \epsilon_{IJKL} \theta_I^2 \theta_{Ja} \theta_{Kb} \theta_{Lc} \Xi^{I,abc}(y) + \frac{1}{2} \sum_{I \neq J \neq K} \theta_{Ia} \theta_J^2 \theta_K^2 \Xi^{I(JK),a}(y) \\
& + \frac{1}{2} \sum_{\substack{I<J \\ I \neq J \neq K \neq L}} \theta_{Ia} \theta_{Jb} \theta_K^2 \theta_L^2 C^{IJ(KL),ab}(y) + \frac{1}{6} \sum_{I \neq J \neq K} \theta_I^2 \theta_J^2 \theta_K^2 C^{(IJK)}(y) \\
& + \frac{1}{6} \sum_{I \neq J \neq K \neq L} \theta_{Ia} \theta_J^2 \theta_K^2 \theta_L^2 \Omega^{I,a}(y) + \theta_1^2 \theta_2^2 \theta_3^2 \theta_4^2 D(y).
\end{aligned} \tag{D.5}$$

The fractional coefficients are such that each field appears only once in the sum. The four-dimensional components are expressed in terms of the six-dimensional ones appearing in (D.4). For example, for the

first few components we have:

$$\begin{aligned}
A(y) &= (X^+)^{\Delta} A(X, 0) \\
\Lambda^{Ia}(y) &= (X^+)^{\Delta+1} \left[\lambda^{Ia}(X, 0) - i y_{\mu} (\bar{\sigma}^{\mu})^{\dot{a}a} \lambda_{\dot{a}}^I(X, 0) \right] \\
F^{IJ,ab}(y) &= (X^+)^{\Delta+1} \left[i X_{\alpha\beta} \epsilon^{ab} E_{\text{aver.}}^{IJ,\alpha\beta}(X, 0) - 4i \epsilon^{ab} \frac{\partial A_{\text{aver.}}}{\partial \varphi_{IJ}}(X, 0) \right] \\
F^I(y) &= (X^+)^{\Delta+1} \left[-i X_{\alpha\beta} E^{I,\alpha\beta}(X, 0) + 2i \frac{\partial A}{\partial \varphi_{II}}(X, 0) \right] \\
\Upsilon^{I,abc}(y) &= (X^+)^{\Delta+2} \left[\epsilon^{ab} i X_{\alpha\beta} \chi_{\text{aver.}}^{I,\alpha\beta c}(X, 0) + \epsilon^{ab} X^{\alpha\beta} y_{\mu} (\bar{\sigma}^{\mu})^{\dot{a}c} \epsilon_{\dot{a}\dot{c}} \chi_{\text{aver.}}^{I,\alpha\beta\dot{c}}(X, 0) \right. \\
&\quad \left. - 24i \epsilon^{bc} \epsilon_{IJKL} \frac{\partial \lambda^{Ja}}{\partial \varphi_{KL}}(X, 0) - 24 \epsilon_{IJKL} \epsilon^{bc} y_{\mu} (\bar{\sigma}^{\mu})^{\dot{a}a} \frac{\partial \lambda_{\dot{a}}^J}{\partial \varphi_{KL}}(X, 0) \right] \\
\Upsilon^{I,a}(y) &= (X^+)^{\Delta+2} \left[-i X_{\alpha\beta} \chi^{IJ,\alpha\beta a}(X, 0) - X_{\alpha\beta} y_{\mu} (\bar{\sigma}^{\mu})^{\dot{c}a} \epsilon_{\dot{c}\dot{d}} \chi^{IJ,\dot{d}}(X, 0) + 2i \frac{\partial \lambda^{Ia}}{\partial \varphi_{IJ}}(X, 0) \right. \\
&\quad \left. + y_{\mu} (\bar{\sigma}^{\mu})^{\dot{a}a} \frac{\partial \lambda_{\dot{a}}^I}{\partial \varphi_{JJ}}(X, 0) - i \frac{\partial \lambda^{Ja}}{\partial \varphi_{IJ}}(X, 0) - y_{\mu} (\bar{\sigma}^{\mu})^{\dot{a}a} \frac{\partial \lambda_{\dot{a}}^J}{\partial \varphi_{IJ}}(X, 0) \right] \\
\ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{aligned} \tag{D.6}$$

etc. Here the averaged quantities are defined whenever $I \neq J$ as done in the main text, e.g.

$$\theta_{I\alpha} \theta_{J\beta} E^{IJ,\alpha\beta}(X, \varphi) = -i X_{\alpha\beta} \theta_I \cdot \theta_J E_{\text{aver.}}^{IJ,\alpha\beta}(X, \varphi), \tag{D.7}$$

and should be regarded as shorter labels of longer expressions.

Let us now count the number of component fields contained in the multiplet (D.5). There are:

- one scalar field $A(y)$
- four spinors $\Lambda^I(y)$ in the representation $(1/2, 0)$
- $3 \times 4/2 = 6$ rank-2 tensors $F^{IJ}(y)$, $I < J$ in the representation $(1, 0)$
- four scalars $F^I(y)$
- four spinors $\Upsilon^I(y)$ in the representation $(3/2, 0)$
- $4 \times 3 = 12$ spinors $\Upsilon^{IJ}(y)$, $I \neq J$, in the representation $(1/2, 0)$
- one rank-4 tensor $B(y)$ in the representation $(2, 0)$
- $4 \times 3 = 12$ rank-2 tensors $B^{JKL}(y)$, $J < K$ and $J \neq K \neq L$, in the representation $(1, 0)$
- $4 \times 3/2 = 6$ scalars $B^{(IJ)}(y)$, $I < J$ and symmetric in I, J
- four spinors $\Xi^I(y)$ in the representation $(3/2, 0)$
- $4 \times (3 \times 2/2) = 12$ spinors $\Xi^{I(JK)}$, $J \neq K \neq L$ and symmetric in K, L , in the representation $(1/2, 0)$
- six rank-2 tensors $C^{IJ(KL)}(y)$, $I < J$ and symmetric in K, L , in the representation $(1, 0)$
- $4 \times 3 \times 2/3! = 4$ scalars $C^{(IJK)}(y)$, symmetric in I, J, K
- four spinors $\Omega^I(y)$, in the representation $(1/2, 0)$
- one scalar $D(y)$.

This field content can be summarized by using the Pascal pyramid at layer $\mathcal{N} = 4$:

$$\begin{array}{cccccccccccccccc}
1 & & 4 & & 6 & & 4 & & 1 & & A & & \Lambda^I & & F^{IJ} & & F^I & & B \\
& 4 & & 12 & & 12 & & 4 & & & & \Upsilon^I & & \Upsilon^{IJ} & & B^{JKL} & & \Xi^I & \\
& & 6 & & 12 & & 6 & & & \Longleftrightarrow & & B^{(IJ)} & & \Xi^{I(JK)} & & C^{IJ(KL)} & & \\
& & & 4 & & 4 & & & & & & C^{(IJK)} & & D & & \Omega^I & & \\
& & & & 1 & & & & & & & & & & & & &
\end{array} \tag{D.8}$$

The total number of fields is given by the sum of the elements of the pyramid and amounts, as it should be, to $3^4 = 81$.

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